LH - SPACES, TRANSFORMATIONS AND DISTRIBUTIONS

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पूज्य पिताजी, माताजी तथा बड़े भाईयों

के

श्री चरणों

में

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CERTIFICATE

This is to certify that the research work embodied in the present dissertation entitled ''LH-Spaces, Transformations and Distributions'' by A.K.Tiwari, a Ph.D. scholar of this Department, has been carried out under our supervision and that it has not been submitted elsewhere for any degree or diploma.

(P.K.Kamthan)

M Gupta
23/7/84
(M.Gupta)

July 1984

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A.K.Tiwari

CONTENTS

SYNOPSIS

The theory of generalized integral transforms (distributional integral transforms) has its origin in the work of Schwartz who first extended Fourier transforms to generalized functions. After Schwartz many mathematicians namely Gelfand and Shilov, Zemanian etc. have worked in this direction. Gelfand and Shilov introduced spaces of type S, discussed several properties of these spaces and used them to solve the Cauchy problem. Lee introduced spaces of type H_μ, discussed its several properties, defined Hankel transforms for distributions. He used these spaces for the study of Dirichlet problem in cylindrical coordinates. Fisher obtained several results on the divergent products of distributions and also obtained some results on δ distributions by making the change of independent variable.

Motivated from the work of Gelfand and Shilov, Lee, Fisher and Zemanian, we have introduced in the present thesis spaces of type LH and have discussed several properties including structural properties of these spaces. We have also given applications of these spaces in solving a partial differential equation (in cylindrical coordinates) arising in propagation of heat, relative to two different sets of boundary conditions given in terms of generalized functions.

This dissertation entitled ''LH-Spaces, Transformations and Distributions'', contains six chapters.

Chapter 1 is a collection of some definitions and known results without proof from different texts, research papers, monographs and dissertations on the theory of topological vector spaces and the duality theory, distributions, operators, test function.. spaces and distributional integral transforms. These results are to be used in the subsequent work of this thesis.

Chapter 2 sketches a brief history of the development of delta function, Heaviside function, generalized functions, integral transformations, operational calculus and generalized integral transformations.

Chapter 3 is devoted to the systematic study of several different type of LH-spaces consisting of exponentially decreasing infinitely differentiable functions defined on the first and second quadrant of the plane. This study includes interrelationship among these spaces, construction of several examples showing their non-trivial character and the strict inductive limits of these spaces.

Chapter 4 deals with the continuity property of various types of operators, namely, dilation, multiplier, differential and integral on spaces of type LH. In particular, the conventional Laplace-Hankel transform from a space of type

LH into another space of type LH is also shown to be continuous. Further, this chapter also incorporates the structural study of spaces containing the Laplace-Hankel transforms of members of LH-spaces.

Chapter 5 incorporates the study of impulsive distributions (linear combination of δ-distributions and their derivatives) which are obtained by extracting finite part from divergent integrals. The change of the independent variable has been used to define some impulsive distributions. Finally, Laplace-Hankel transforms of these impulsive distributions have also been investigated.

Chapter 6 includes the applications of spaces of type LH and Laplace-Hankel transform in finding the solution of a partial differential equation arising in the propagation of heat transform, relative to two different sets of boundary conditions given in terms of generalized functions. This equation is considered in cylindrical coordinates with second coordinate θ being kept constant.

CHAPTER - 1

PREREQUISITES

CONTENTS

1. INTRODUCTION AND NOTATION :

The theory of spaces of the type LH presented in subsequent chapters depends essentially upon various results from topological vector spaces, distributions, integral transforms and their ramifications.

Therefore, as a prelude to our work, we list here all those important results from the theory of topological vector spaces, distributions and integral transforms we are going to make ample use of in the Chapters 3,4,5 and 6. These results are taken from several monographs, research papers, and standard texts; see, for example, Refs. [1], [30], [33], [37], [42], [48], [53], [55], [64], [65], [90], [131] and [136] and many other research papers and thesescited throughout the chapters. All results in this chapter are stated without proof and can be found in atleast one of the sources mentioned above. In case a particular result is not to be found in any of the standard texts, the corresponding reference is given either before or after the statement.

Throughout the sequel we use the following notation :

$\mathbb{N}$ = set of all positive integers

$\mathbb{R}$ = set of all real numbers

$\mathbb{C}$ = set of all complex numbers

$\mathbb{R}_+ = (0, \infty)$

$\mathbb{R}_- = (-\infty, 0)$

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ equipped with its usual topology

$\mathbb{R}^n$ = n-dimensional real euclidean space

$\mathbb{C}^n$ = n-dimensional complex euclidean space

Ω = an open subset in $\mathbb{R}^n$

X = a nontrivial vector space over $\mathbb{K}$

Also the symbol # at the end of a proof indicates its completion.

2. TOPOLOGICAL VECTOR SPACES AND DUALITY :

A topological vector space (abbreviated TVS) is a pair (X,T) of a vector space X and a linear topology T. We denote by $\mathcal{B}_T$ or $\mathcal{B}$ (if there is no confusion likely to arise by dropping the letter T) the fundamental neighborhood system at origin consisting of balanced, absorbing sets such that for $u \in \mathcal{B}$ there is $v \in \mathcal{B}$ with $v+v \subseteq u$. If each $u \in \mathcal{B}$ is also convex, T is a locally convex topology (l.c. topology) and in that case we call (X,T) a locally convex topological vector space (abbreviated l.c.TVS). The symbol $D \equiv D_T$ stands for the family of all seminorms generating the locally convex topology T. If Y is a subspace of (X,T), the induced topology on Y from T is denoted by $T|Y$. We assume throughout this work that the topology T is Hausdorff.

We recall from ([48], p. 98) the following useful

PROPOSITION 2.1 : Let X be a vector space and let T_1 and T_2

be two l.c. topologies on X, where T_1 is generated by the family D_{T_1} of seminorms and T_2 is generated by the family D_{T_2} of seminorms. Then T_1 is finer than T_2 if and only if for each $\rho \in D_{T_2}$ there exists a $\gamma \in D_{T_1}$ and $M \in \mathbb{R}_+$ such that

$$\rho(x) \leq M\gamma(x), \quad \forall\, x \in X.$$

A TVS (X,T) is said to be <u>complete</u> (respectively <u>sequentially complete</u>) if every Cauchy net (respectively Cauchy sequence) converges in (X,T). (X,T) is a <u>Fréchet space</u> if it is a complete, metrizable l.c.TVS. Let us mention here that an l.c. topology is metrizable if and only if it is generated by a countable family of seminorms. If this countable family consists of norms, we call the space a <u>countably multinormed space</u>.

<u>DUALITY</u> :

Let us mention at the outset that we use the symbol X' and X^* respectively for the algebraic and topological duals of an l.c.TVS X.

Coming to the general theory of duality, let us denote by $\langle X,Y\rangle$ a dual pair of two vector spaces X and Y defined over the same field $\mathbb{K}$; that is, there exists a bilinear form $B: X \times Y \to \mathbb{K}$, $B(x,y) \equiv \langle x,y\rangle$, which separates points of X as well as of Y. The weak, strong and Mackey topologies on X respectively denoted by $\sigma(X,Y)$, $\beta(X,Y)$ and $\tau(X,Y)$ are generated by the families $\{\gamma_A : A \in S\}$ where for $x \in X$, $\gamma_A(x) = \sup_{y \in A} |\langle x,y\rangle|$ and S

respectively denote the family of all finite subsets of Y, the family of all $\sigma(Y,X)$ bounded subsets of Y and the family of all balanced, convex and $\sigma(Y,X)$-compact subsets of Y.

INDUCTIVE LIMITS :

In this subsection, we shall briefly touch upon the results and notions involoving inductive limits. To begin with, we have

DEFINITION 2.2 : Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of l.c.TVS and R_α a linear map from X_α to X, where $X = \bigcup_{\alpha \in \Lambda} \{R_\alpha[X_\alpha]\}$. Then the finest l.c. topology T on the vector space X such that each R_α is continuous on X is called the inductive limit of X_α. In case $\Lambda = \mathbb{N}$, (X,T) is known as the generalized inductive limit of $X_n (n \in \mathbb{N})$, and the topology T is called the generalized inductive limit topology.

PROPOSITION 2.3 : Let (X,T) be the generalized inductive limit of (X_n, T_n), $X_n \subset X_{n+1}$, $n \in \mathbb{N}$, $X = \bigcup_{n \geq 1} X_n$, with respect to injective maps $I_n : X_n \to X$. Assume that $S_n = T_{n+1} | X_n$, is equivalent to T_n for each $n \geq 1$. Then T induces on each X_n the topology T_n.

DEFINITION 2.4 : If the generalized inductive limit topology T on $X = \bigcup_{n \geq 1} X_n$ with respect to injective maps $I_n : X_n \to X$ induces on each X_n the topology equivalent to its original topology T_n, then T is called the strict inductive limit

<u>topology</u> and the pair (X,T) is called the <u>strict inductive limit</u> of $\{X_n\}$.

The following result is due to Dieudonne' and Schwartz.

<u>THEOREM</u> 2.5 : Let (X,T) be the strict inductive limit of the l.c. TVS $\{X_n,T_n\}$. Assume that X_n is T_{n+1}-closed in X_{n+1} $(n \geq 1)$. Then a set $B \subset X$ is T-bounded in X if and only if $B \subset X_n$ and B is T_n-bounded for some $n \in \mathbb{N}$; also, (X,T) is complete if and only if for every Cauchy net $\{x_\alpha\}$ in X_n relative to the induced topology τ_n on X_n from T there exists $p \in \mathbb{N}$, $p \geq n$, such that $\{x_\alpha\}$ is convergent in (X_p,τ_p), and in turn it follows that (X,T) is complete if and only if each (X_n,T_n) is complete.

3. <u>LINEAR AND ADJOINT OPERATORS</u> :

In this section we collect some basic results on linear and adjoint operators, to be used in the subsequent work of this thesis.

To begin with, let us recall the notion of an <u>isomorphism</u> which means a bijective linear map R from one TVS (X,T_1) onto another TVS (Y,T_2) such that the map R and the inverse map R^{-1} are continuous.

A frequently used characterization of a linear map is contained in

<u>PROPOSITION</u> 3.1 : A linear map $R : (X,T_1) \rightarrow (Y,T_2)$ is continuous

if to each $\rho \in D_{T_2}$ there exists $\gamma \in D_{T_1}$ and $M \in \mathbb{R}_+$ such that

$$\rho\{R(x)\} \leq M\gamma(x), \quad \forall\, x \in X.$$

Next we have

PROPOSITION 3.2 : Let X be a vector space, $\{X_i\}_{i \in \Lambda}$ a family of locally convex spaces, and R_i linear maps from X_i into X, for each i in Λ. Further assume that T is the finest locally convex topology on X for which all the maps R_i are continuous. If g is a linear map from X into another l.c. TVS Y, then g is continuous if and only if all the maps $g \circ R_i : X_i \to Y$ are continuous.

ADJOINT OPERATORS :

Let X and Y be two vector spaces and R a linear map from X to Y. We denote by R′ the transpose of R defined from Y′ to X′ as follows

$$< R(x), y' > \;=\; < x, R'(y') >, \quad x \in X,\ y' \in Y'$$

If X and Y are TVS, the restriction of R′ to Y^*, is written as R^*, i.e. $R^* = R'|Y^*$; $R^*(Y^*) \subset X^*$ if R is continuous. However, in this direction more information is contained in

PROPOSITION 3.3: Let (X,T_1) and (Y,T_2) be two l.c. TVS. If the linear map $R: X \to Y$ is T_1–T_2 continuous, then $R^* : Y^* \to X^*$ is $\sigma(Y^*,Y)$–$\sigma(X^*,X)$ continuous and also $\beta(Y^*,Y)$–$\beta(X^*,X)$

continuous. Furthermore, if R is an isomorphism from X onto Y, then R^* is a $\sigma(Y^*,Y)$-$\sigma(X^*,X)$ isomorphism from Y^* onto X^*.

4. TEST FUNCTION SPACES :

This section incorporates basic results on four types of spaces of test functions useful in the sequel. Let us first introduce

DEFINITION 4.1 : A function f is said to be (i) conventional if it is a real or complex valued function defined on a subset of $\mathbb{R}^n$ or $\mathbb{C}^n$; (ii) locally integrable on Ω if it is a conventional function, Lebesgue integrable on every open set A in $\mathbb{R}^n$, such that $\bar{A}$ is a compact subset of Ω ; (iii) of rapid descent if it is a conventional function on $\mathbb{R}^n$ or $\mathbb{C}^n$ and

$$|f(x)| = o(|x|^{-m}) \text{ as } |x| \to \infty ,$$

for every integer m in $\mathbb{N}$; and (iv) smooth if it is a conventional infinitely differentiable function such that its derivatives of all orders are continuous at all points of its domain, (v) The support of a continuous conventional function f defined on Ω is the closure of the set $\{x \in \Omega : f(x) \neq 0\}$ in Ω. We write it as supp f.

Coming to the test function spaces, we have

DEFINITION 4.2 : (i) The vector space of all smooth functions

defined on Ω is designated as $E(\Omega)$; (ii) For a compact subset K of $\mathbb{R}^n$ with $K \subset \Omega$, we denote by D(K) the subspace of $E(\Omega)$ such that each member of D(K) has support contained in K (iii) The symbol S stands for the vector space of all rapidly decreasing functions defined on $\mathbb{R}^n$, which possess continuous rapidly decreasing partial derivatives of all orders.

We equip the spaces $E(\Omega)$, D(K) and S with Fréchet-topologies T_E, T_K , T_S respectively generated by the families

$$\mathfrak{S}_E = \{ \gamma_{K,p} : \text{K varies over compact subset of } \Omega \text{ and } p \in \mathbb{N}^n \},$$

$$\mathfrak{S}_K = \{ \rho_p : p \in \mathbb{N}^n \}$$

and

$$\mathfrak{S}_S = \{ \lambda_{n,p} : n \in \mathbb{N},\ p \in \mathbb{N}^n \}$$

of seminorms, where

$$\gamma_{K,p}(\varphi) = \sup_{x \in K} |D^p \varphi(x)| ,$$

$$\rho_p(\varphi) = \sup_{x \in K} |D^p \varphi(x)|$$

and

$$\lambda_{n,p}(\varphi) = \sup_{x \in \mathbb{R}^n} |(1+|x|^2)^n D^p \varphi(x)|.$$

Let us observe that the members of D(K) can be extended to the whole space $\mathbb{R}^n$, by assigning zero to each point

outside K.

If $\{K_n : n \geq 1\}$ is an increasing sequence of compact subsets of Ω such that each compact subset of Ω is contained in some K_n ; then clearly $D(K_n) \subset D(K_{n+1})$, $n \geq 1$. Define

$$D(\Omega) = \bigcup_{n \in \mathbb{N}} D(K_n).$$

The space $D(\Omega)$ equipped with the inductive limit topology T defined with the help of injective maps from $D(K_n)$ to $D(\Omega)$ is known as the space of test functions with compact support contained in Ω. If $\Omega = \mathbb{R}^n$, we write D for $D(\Omega)$

Let us recall from ([48], p.169) the following useful

PROPOSITION 4.3 : Let A be a closed subset of Ω , and V an open subset of Ω containing A. Then there exists an infinitely differentiable function φ such that $0 \leq \varphi(x) \leq 1$ for $x \in \Omega$, $\varphi(x) = 1$ for $x \in A$, and $\varphi(x) = 0$ for $V^c = \Omega \setminus V$.

Finally in this section, we have

DEFINITION 4.4 : Let Ω denote the open interval (o,∞). For each real number μ the space H_μ is defined as

$$H_\mu = \{ \varphi : \varphi \in E(\Omega),$$

$$\gamma^{\mu}_{k,q}(\varphi) = \sup_{\Omega} |x^k (x^{-1}D)^q \{ x^{-(\frac{2\mu+1}{2})} \varphi(x)\}| < \infty \ ; k, q = 0,1,\ldots \}.$$

The space H_μ is equipped with Fréchet-topology T_{H_μ} generated

by the family $\{\gamma^{\mu}_{k,q}\}_{k,q=0}^{\infty}$ of seminorms.

5. DISTRIBUTIONS :

This section includes results on distributions which are defined as continuous linear functionals on the Fréchet space $D(\Omega)$ of test functions introduced in the preceding section. Indeed, after having introduced the notions of singular, regular distributions, differentiation of distributions, we pass on to defining product of distributions and the association of distributions on a given class of smooth functions with distributions on a class of test functions obtained from the given class by change of the independent variable. To begin with we have

DEFINITION 5.1 : Let f be a locally integrable function defined on Ω. A distribution T_f generated by the relation

$$< T_f, \varphi > = \int_{\mathbb{R}^n} f(x)\, \varphi(x)\, dx$$

is known as regular distribution. A distribution which is not regular is known as singular distribution. An example of a singular distribution is δ distribution defined by $< \delta, \varphi > = \varphi(0)$.

Concerning the δ-distribution, let us note that it is a limiting case of a family of regular distributions.

Indeed we have [48, p.316] :

THEOREM 5.2 : Let ρ be a function satisfying the following properties :

i) ρ is infinitely differentiable in $\mathbb{R}^n$

ii) support $\rho = B_1$ where $B_1 = \{x: |x| \leq 1\}$

iii) $\rho(x) \geq 0$ for all $x \in \mathbb{R}^n$

iv) $\int_{\mathbb{R}^n} \rho(x)dx = 1.$

If $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon})$, then $\{\rho_\varepsilon\}$ converges to δ in $\sigma(D^*(\mathbb{R}^n), D(\mathbb{R}^n))$ and also in $\beta(D^*(\mathbb{R}^n), D(\mathbb{R}^n))$.

Denoting by H the Heaviside function $[H(x) = 1,\ x \geq 0$ and $= 0,\ x < 0\ ]$, let us recall from ([33], p.106) the following

<u>THEOREM</u> 5.3 : Let ρ be the function as defined in Theorem 5.2 write

$$\delta_n(x) = n\rho(nx)$$

and

$$H_n(x) = \int_{-\frac{1}{n}}^{\frac{1}{n}} H(x-t)\delta_n(t)dt.$$

Then

$$\frac{d}{dx}\{H_n(x)\} = \delta_n(x).$$

Next we introduce

<u>DEFINITION</u> 5.4 : Let Ω be an open subset of $\mathbb{R}^n$ and $p \in \mathbb{N}^n$. For $T \in D^*(\Omega)$, <u>the partial derivative</u> $\partial^p T \equiv T^p$ of index p of T is defined by

$$\langle T^p, \varphi \rangle \equiv \langle \partial^p T, \varphi \rangle = (-1)^{|p|} \langle T, \partial^p \varphi \rangle, \quad \partial^p \equiv \frac{\partial^{p_1+p_2+\dots+p_n}}{\partial x_1^{p_1} \dots \partial x_n^{p_n}}$$

DEFINITION 5.5 : An impulsive distribution is a linear combination of δ distribution and its derivatives . It is known in the literature ([48], p.324) that the derivative of the distribution T_H generated by the Heaviside function H is nothing but the δ-distribution, that is, $H' = \delta$.

Coming to the product of distributions, let us observe that if two distributions T_f and T_g are regular, then their product T_h corresponds to the distribution defined by locally integrable function h, where $h(x) = f(x)g(x)$.

However, the product of a regular distribution T_f with delta distribution δ is defined in

DEFINITION 5.6 : Let T_f be a distribution defined on $D(\Omega)$ where Ω is an open subset of $\mathbb{R}$. For ρ defined in Theorem 5.2, set

$$\delta_n(x) = n\rho(nx)$$

and

$$f_n(x) = \int_{-\frac{1}{n}}^{\frac{1}{n}} f(x-t)\delta_n(t)dt.$$

We say that the product $T_f . \delta$ of T_f and δ exists if $\{f_n\delta_n\}$ converges weakly to distribution S in $D^*(\Omega)$. We define the product $T_f.\delta$ as S.

For the change of independent variable we follow [131] and pass on to

DEFINITION 5.7 : Let (a,b) and (c,d) be two open intervals and $g : (a,b) \to (c,d)$ is an infinitely differentiable one-one onto

function such that the inverse h of g has nonzero derivative. Then corresponding to a distribution T in $D^*(c,d)$ there exists a unique distribution S in $D^*(a,b)$ as follows :

$$\langle S,\varphi\rangle = \langle T, |h'| .\varphi \circ h\rangle$$

for $\varphi \in D(a,b)$, where $f \cdot g$ means product of f with g and fog means composition of f and g.

6. DISTRIBUTIONAL INTEGRAL TRANSFORMS :

This section singles out just two definitions from [136] which we will use in Chapters 5 and 6.

DEFINITION 6.1 : A distributional Laplace transform l^* is a linear functional from $D^*(\mathbb{R})$ to $\mathbb{K}$, defined by

$$\langle l^* f, \Phi\rangle = 2\pi i \langle f, \check{\varphi}\rangle$$

where $\Phi(s) = l(\varphi)$, $f \in D^*(\mathbb{R})$ and $\check{\varphi}(t) = \varphi(-t) \in D(\mathbb{R})$

DEFINIITION 6.2 : A distributional Hankel transform h_μ^*, $\mu \geq -\frac{1}{2}$ is a linear functional from H_μ^* to $\mathbb{K}$, given by

$$\langle h_\mu^* f, \Phi\rangle = \langle f,\varphi\rangle$$

where $\Phi(y) = h_\mu(\varphi(x))$, $f \in H_\mu^*$ and $\varphi(x) \in H_\mu$.

7. OTHER USEFUL RESULTS :

We first consider two different spaces of smooth functions contained in [136]

DEFINITION 7.1 : The space (H) is the vector space of all

smooth functions $\theta(t)$ on $-\infty < t < \infty$ such that for each nonnegative integer k there exists an integer N_k for which $(1+t^2)^{-N_k} \cdot D^k \theta(t)$ is bounded on $-\infty < t < \infty$.

DEFINITION 7.2 : The linear space J consists of all smooth functions $\Psi(x)$ defined on $0 < x < \infty$ such that for each nonnegative integer p, there is an integer N_p for which $(1+x^{N_p})^{-1}(x^{-1}D)^p\Psi(x)$ is bounded on $0 < x < \infty$.

We also need (cf. [1])

PROPOSITION 7.3 : Let $\{f_n\}$ be a sequence of real valued functions having finite derivative at each point of an open interval (a,b). Assume that for atleast one point x_o in (a,b) the sequence $\{f_n(x_o)\}$ converges. Also suppose that there exists a function g such that $f_n' \to g$ uniformly on (a,b). Then

(1) There exist a function f such that $f_n \to f$ uniformly on (a,b)

(2) For each x in (a,b) the derivative $f'(x)$ exists and equals $g(x)$.

CHAPTER -2

DEVELOPMENT

CONTENTS

1. THE DELTA FUNCTION :

The delta (δ) function well known to physicists as well as mathematicians with the name of impulse function also, was introduced by Dirac around the year 1920 while working on some quantum mechanical problems in the form of

$$\delta(x) = 0, \ x \neq 0 \text{ and } \int \delta(x) \ \varphi(x)dx = \varphi(0)$$

where φ is a continuous function. Indeed, it is defined by the following properties:

$$\delta(x) = 0, \ x \neq 0; \ \delta(0) = \infty, \ \int_{-\infty}^{\infty}\delta(x)dx = 1.$$

Dirac himself pointed out that δ is not a function in the classical sense ; but it can be regarded as a limit of certain sequence of functions, for example, see [77], [124], [131] and also Theorem 1.5.2.

Though the δ -function cannot be defined as a function in the sense of ordinary analysis, but its applications are varied in nature. If one looks into monographs, books or papers on mathematical physics, generalized functions and especially the problems involving jumps or translations, one finds the δ-function appearing in most of the cases. Indeed, this function and its distributional derivatives are found commonly used in the analysis of impulses, concentrated loads, other point singularities such as electrostatic charges, and various boundary value problems of mechanics and mathematical physics [77].

Before we pass on to mention a few words about the properties of δ- function, let us recall another very simple function H known as the unit step function or Heaviside function (cf. Theorem 1.5.2) appeared in the literature long before the existence of δ-function. Indeed, its indirect use dates back to 19th century by Cauchy [14], Poisson [87] and Kirchhoff [60]. However, its present form is due to Heaviside [43] who introduced the same in 1893 and suggested that the derivative of Heaviside function is nothing but the δ -function [77].

The inconsistency of the definition of δ-function with the theory of Lebesgue integration is obvious, for the integral of an almost everywhere zero function is nonzero. However, as pointed out earlier this unusual behaviour of δ-function leads to the interesting property of translation, namely

$$\int_{-\infty}^{\infty} \delta(x)h(t-x)dx = \int_{-\infty}^{\infty} \delta(t-x)h(x)dx = h(t)$$

where h(t) belongs to a large class of functions having some mild conditions, e.g. integrability. An interesting consequence of the shifting property of the δ-function is the evaluation of its Laplace transform ; indeed,

$$L\delta(t) = \int_{0}^{\infty} e^{-st}\,\delta(t)\,dt = 1$$

Aroung the year 1958, Maurice [73] used the Dirac

delta function to express the boundary conditions for a subspace S derived from Maxwell's equation for the elctric and magnetic potentials, and applied the same to current systems for a special type of surface S, namely sphere. In 1972, the fractional powers of δ-function like $\delta^{1/2}$ and $\delta^{5/6}$ etc., too were not spared by Thurber and Katz [117] who used them to give meaning to several expresssions involved in the electromagnetic theory, quantum scattering theory and perturbation expansions in quantum field theory. However, in order to avoid unresolvable inconsistencies the use of these notions were based on the nonstandard analysis of A. Robinson [91].

On the other hand, Keiko [59] in the year 1981 defined the δ-function from the point of integration theory and proved some of its elementary properties. He evolved a process of differentiation which he applied to differentiate the δ- function.

Almost a year later in 1982, Raju [89] introduced pointwise product and composition of distributions with δ -functions, the details of which we shall take up in the next section. Indeed, all the properties of the delta function discussed above as well as the δ-functions themselves and their derivatives have much to do in ''The theory of distributions'' developed by L.Schwartz. However, for an interesting account

of history of δ-function we refer to [20], [72], [77].

2. GENERALIZED FUNCTIONS:

As the name suggests, the generalized function or distribution is a generalization of the classical concept of a mathematical function. The origin of this concept is attributed to the work of Wiener who in the process of obtaining solutions of some partial differential equations of functions which are not differentiable, introduced the notions of generalized functions and generalized derivatives. However, functional analysis approach to this concept in extending the notion of a function was given by Sobolev [107] and Schwartz [98]. Indeed, it is said that Sobolev invented distributions, but it was Schwartz who founded the systematic theory of distributions. No doubt, much is known and written about distributions, a good account of its prehistoric development is available in [72] and [105]; yet for getting ourselves more involved in this topic we mention in brief some salient features of its evolution.

There are several extensions and generalizations of generalized or distributional derivatives. Let us consider them one by one in chronological order.

It seems it was DuBois Reymond [23] who first considered in 1879, the infinitely differentiable function φ with φ

and all its derivatives vanishing outside a bounded set in $\mathbb{R}$, while proving the fundamental lemma of calculus of variation, namely "$\int_a^b f(x)\,\varphi(x)\,dx = 0$ for an integrable function f, implies $f(x) = 0$ on $[a,b]$". He also proved, "$\int_a^b f'(x)\varphi(x)\,dx = 0$ for an integrable function f implies f = constant". Though DuBois Reymond proved these results from the point of view of differentiability requirement of the Euler-Lagrange equations of the calculus of variation ; but he did introduce two methods which are basic in distribution theory. Indeed, in modern language of distributions one derives from his second result that f is constant if its generalized derivative is zero.

Next comes the name of Wiener [129] who in 1926 considered generalized solution of second order partial differential equation of the type

$$\rho u \equiv A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = 0$$

where A, B, ..., F are functions in some class C^n.

According to him, if $\rho u = 0$ is an equation of a vibrating string, we regard u as a solution of our differential equation in general sense ; in other words, u is a solution of the equation even if u may not have derivatives of orders appearing in the equation or may not be differentiable at all. In this situation, to find solution of the partial differential equation

of vibrating string, he applied rigorously test function method. Indeed, if $G(x,y)$ is positive infinitely differentiable function which alongwith its derivatives vanish outside a certain region R of XY plane and also on the boundary of R, then he showed that integration by parts yields a function $G_1(x,y)$ of the above nature such that

$$\iint_R (\rho u) G(x,y)\,dxdy = \iint_R u(x,y) G_1(x,y)\,dxdy$$

Thus u satisfies our differential equation $\rho u = 0$ if

$$\iint_R u(x,y) G_1(x,y)\,dxdy = 0$$

In otherwords, a function u which is orthogonal to every such function G_1 is referred to the generalized solution of partial differential equation $\rho u = 0$.

A year later in 1927, Wiener [130] chose to use a type of test surface method for solving partial differential equation, which was different from his earlier work, thereby showing the lack of unified theory for getting generalized solution of such partial differential equation at that time.

To overcome this difficulty, around 1936 Sobolev studied solutions of partial differential equations in function space which gave birth to the modern theory of generalized functions. In fact while dealing with the Cauchy problem for the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 ,$$

there was found a discrepancy between physical data and its mathematical model which could be removed by using the concept of generalized solution. In a series of papers on the Cauchy problem for hyperbolic equation published during 1933 to 1936 Sobolev established the existence and uniqueness of solutions in spaces of generalized functions. These papers not only played an important role in the development of the modern theory of partial differential equations and of generalized functions, but also led him to introduce spaces W_p^l of functions, known as Sobolev spaces these days. Indeed, by a generalized derivative of a locally integrable function f, he meant a locally integrable function g (both f and g being defined in a domain $G \subset \mathbb{R}^n$) satisfying the condition

$$\int_G f(x) D^q \varphi(x)dx = (-1)^{|q|} \int_G g(x)\varphi(x)dx$$

where φ is a function of bounded support and $q=(q_1,\ldots,q_n)$ with $|q| = q_1+q_2+\ldots+q_n$. Using this concept, for $p \geq 1$, $l \geq 0$ Sobolev made extensive study of the space W_p^l which consists of pth power integrable functions f having all generalized derivative $D^q f$ for $|q| \leq l$ in the domain G. Indeed, he found a criterion for the equivalence of various norms possible to define on the space W_p^l, proved the Banach character of the space W_p^l, embedding theorems etc. He used these

embedding theorems to relate the generalized solutions of Cauchy problem with classical ones. However, without going into the details of his investigations in this direction, let us divert our attention to his investigation in distribution theory. In 1936, Sobolev [107] studied elementary properties of distributions, dealt with convergence and differentiation in the space $(D^m)^*$ and D^* as given in Chapter 1 as well as used linear functionals in place of functions and generalized differentiation. Unfortunately, Sobolev did nothing further, using these concepts.

As mentioned earlier, the theory of distribution was extensively nourished by a French mathematician. L. Schwartz during a period of five years from 1945 to 1950 in [95], [96], [97] and [98]. On the basis of earlier theory of generalised derivatives and generalized solution of partial differential equation and the theory of locally convex topological vector spaces, he put forward a systematic account of the theory of distributions in his celebrated monograph [98,99] entitled ''Théorie des distributions''.

Though Schwartz was not influenced by the Dirac delta function for his researches in distribution theory ; but using his devices, for instance $f(\varphi) = \int_{-\infty}^{\infty} f(x)\, \varphi(x)dx$, and $f^n(\varphi) = f[(-1)^n\, \varphi^n(x)]$, where $\varphi(x)$ is an infinitely differentiable function vanishing outside a finite interval, introduced

in [95] a rigorous treatment to the function δ and its derivatives could be given. Besides , in the same paper he extended his results to n-dimension case and gave applications of these concepts to harmonic functions and Fourier transforms. In 1948, Schwartz [96]defined spherical distribution, also known as tempered distribution, as continuous linear functionals on the suitably topologized locally convex topological vector space S consisting of infinitely differentiable and rapidly decreasing functions φ, of n variable (cf. Definition 1.4.2); and proved several results on these distributions including their Fourier transforms. Two years later in 1950, besides discussing various questions of existence, uniqueness etc. of solutions of certain partial and integro-differential equations from the point of view of distributions in [98], Schwartz illustrated this work on distributions with several examples including the solution of heat and homogeneous wave equation. All these results due to him are available in [98,99]. On the otherhand, Schwartz got one expository article [101] published in 1952 where he talked of the integral representation of bi-linear functionals on spaces of distributions in terms of kernels, of which a general unified theory in the form of nuclear spaces and operators is available in [85].

We now pass on to a related theory namely ''Hadamard Finite part of a divergent integral'' once again developed

while finding the solution of a partial differential equation. Indeed, in 1908 Hadamard [41] while working on a partial differential equation was confronted with divergent integrals which he overcame by introducing the generalized integral called the partie finie. (finite part), (cf. [98] p.41). Indeed, later in 1955, Korevaar [62] continued the study of Hadamard finite part for definite integrals of distributions.

Because of the evolvement of the theory of distributions from partial differential equations, it was clear to applied mathematicians, engineers and physicists, the practical importance of this theory. However, too much abstraction of the same was an hindrance to its use by them. With the aim of simplifying the discoveries of Schwartz for physicists and engineers, Mikusinski and Temple came forward to present this theory in terms of continuous functions and their generalized differentiation.

In fact, Mikusinski, having observed several common features between theory of distributions and operators, defined distributional operators, that is the operator of the form $h^{-\lambda} \frac{f}{l^k}$ where h is a translation operator, l an integral operator, f a function of class C, λ a real number and k a non-negative integer; and then defined a distribution as the limit of a sequence of these distributional operators ; cf. [77] for details.

In 1953, Temple [114] wrote an expository paper on the Schwartz theory of distributions from a weak function point of view and Mikusinski's approach to generalized functions by means of weak convergence. Later in 1955 following Mikusinski, and the Cantors approach of construction of real numbers from rational numbers Temple [115] gave a constructive definition of a generalized function in terms of the ordinary functions. Indeed, he defined a generalized function as an equivalence class of regular sequences, where a regular sequence, according to him, means a sequence $\{ g_m \}$ in C^{∞} such that $(g_m \varphi) \to u \varphi$, for all $\varphi \in D$ and $u\varphi$ is continuous in φ and two regular sequences are said to be equivalent if $(g_m - h_m, \varphi) \to 0$. He developed elementary algebraic properties of generalized functions and showed the equivalence of his theory with those of Schwartz and Mikusinski. He also obtained some applications of this theory to Fourier series and integrals. Further, using his concepts of generalized functions, he later gave a simple proof of Dirichlet principle in [116].

Coming to the chronological growth of various aspects of this theory, we start with Myers who [80] in 1961 defined an L_s-distribution as a map from $\mathfrak{J}$ (= set of all functions e^{zt} where $t \in (-\infty, \infty)$ and z ranges over S, S denoting the strip $\sigma_1 < \text{Re} z < \sigma_2$ of the complex plane) into the field of complex number. Denoting by A the space of all L_s-distributions

over the field of complex numbers, he proved that the space of all Schwartz distributions defined on $(-\infty, \infty)$ is isomorphic to a subspace of A.

The well known five volumes [36], [37], [38], [39] and [40] of Gelfand, Shilov and others, on generalized functions give a systematic account of various aspects of this theory. Whereas the volume I treats an elementary introduction to generalized functions along with its application to various problems in analysis, volume II develops the concepts introduced in I, using the theory of linear topological spaces or in particular the theory of countably normed spaces; for instance in II new types of spaces like S_α, S^β and S_α^β called ''Spaces of type S'' used for the study of Cauchy problem [37] and also in the quantum field theory [18], have been discussed in detail. Indeed, much attention has been paid in this beautiful volume to identify and develop the spaces of the type S, based on the properties of countably normed spaces. However, by an use of the theory of inductive limits of locally convex spaces, many of the results in this volume can be simplified, e.g. [53]. A systematic use of the results obtained in volume II is given in volume III, where they apply the same to the theory of differential equations, giving particular emphasis to Cauchy problem in partial differential equation. Volumes IV and V contain advanced and modern researches on various aspects namely

problems of probability theory related to generalized functions (general random process), but also the problems involving the theory of representation of Lie groups and the kernel theorem of Schwartz.

Around the year 1967, Dacie [19] tackled the problem of homogeneous linear differential equation with meromorphic coefficients to find sufficient conditions for the Dirac distribution δ to satisfy the equation.

Next we cite the name of Fisher who in a series of papers [30], [31], [32], [33] and [34] gave several results on the divergent products of distributions. He obtained these products as linear combination of delta distributions and their derivatives ; and impulsive distributions (cf. Definition 1.5.5) from original ones, by using change of independent variable.

Carmichael and his group [8], [9], [10], [11] from 1973 to 1979 discussed distributions corresponding to tubular radial domains T ($= \mathbb{R}^n + i\, C$, C an open cone in $\mathbb{R}^n$). In [8] Carmichael related functions which are analytic and have a specified growth conditions in T to the Cauchy and Poisson integrals of their distributional boundary values. In [11] Carmichael and walker represented distributions having compact support as the boundary value of Cauchy and Poisson integrals corresponding to tubular radial domains. They extended

the results to vector valued distributions. Carmichel and Milton discussed distributional boundary values in the dual spaces of spaces of types S in [10]. In [9] Carmichael obtained the representations of analytic functions in terms of Fourier-Laplace transform of distributions of exponential growth.

On the other hand McBride developed the theory of fractional calculus for generalized functions and showed its use in solving the dual integral equations of Titchmarsh type [103] (see for instance [74], [76]).

For several applications of generalized functions, in particular of delta-distribution in economics and engineering we refer to the work of Kanwal [58] and Estrada and Kanwal [27], [28].

In 1982, Raju used the products of the type $\delta^2, \delta^{(k)}.x^{-n}$ in quantum field theory and also discussed the products $\delta^{(k)}.x^{2m}$, $\delta(1-\sin x)$. Pugh [88] defined the analytic signal of a generalized function and of a generalized stochastic process. He also discussed L_2-distribution theory and its applications.

Recently Colombeau has introduced a new theory of generalized functions so as to give meaning to any product of distributions in his work [15], [16] and [17]. He has shown that these generalized functions form an algebra with respect to the

multiplication defined by him and this multiplication generalizes exactly the usual multiplication of C^∞ functions defined on an open subset of $\mathbb{R}^n$.

Though, we have omitted many important references playing a key role in the development of distribution theory, for instance Bremmerman [4], Light Hill [68], Szaz [109,110], Tillman [118], etc.; but we have tried to uptodate the recent advances . On the one hand it would be worthwhile to mention here the researches in periodic distributions due to Szaz [110], Estrada and Kanwal [28] ; whereas the present thesis embodies work on product of distributions, finite part of divergent integrals giving rise to impulsive distributions etc.

Notwithstanding the advances made in the theory of distributions, there is something very important yet to be achieved. In fact, it would be worthwhile to put our energy to investigate the existence of Schauder bases (cf. [56]) in most of these test spaces including the spaces of type S, W etc. and then identify these spaces as sequence spaces-in particular the Kothe spaces $\Lambda(P)$; cf [55]. Once this is achieved, it would then be an easy going task to identify the distributions on the spaces in terms of certain simple looking infinite series ; for instance, one may see [54], [57]. In this direction, we may recall the work of Vogt [120].

3. OPERATIONAL METHODS AND INTEGRAL TRANSFORMS :

Treating differentiation and integration as algebraic operations, one finds in ''Operational method'' a procedure for solving differential equations. Its origin dates back to seventeenth century when Gottfried Wilhelm Leibnitz (1646-1716) observed the similarity between the nth differential coefficient of a product of two functions and nth degree binomial. Motivated by the derivatives of the exponential function e^{ax}, he further gave an idea about its fractional derivatives ; indeed, he defined

$$D^p e^{ax} = a^p e^{ax},$$

where p is a positive, negative number or a fraction and $D \equiv \frac{d}{dx}$.

Around 1737, Leonhard Euler tried to make headway in the ideas introduced by Leibnitz for solving differential equations without making significant contribution in this direction ; however, his integral transformation

$$A(\varphi) = \int e^{tx} \varphi(x) \, dx$$

helped him to obtain solutions of certain definite classes of differential equations in [29].

Almost simultaneously, Joseph Louis Lagrange, motivated by the analogy of differentiation and binomial expansion due

to Leibnitz regarded differentiation symbol D as an ordinary algebraic symbol for computation purposes and used these notions for solving difference equations.

In the begining of nineteenth century, Pierre- Simon de Laplace [63] represented transformation introduced by Euler by a function f in the following form

$$f(x) = \int F(p)\ e^{px}\ dp$$

where limits of integration are chosen suitably. He used this to solve differential as well as difference equations.

At the end of nineteenth century, it was an English professional engineer Oliver Heaviside, who further developed the concepts of operational methods and applied these ideas extensively to a number of problems in physics, engineering and applied mathematics. For the first time, the operational calculus was used by him for the solution of the equations of telegraph theory, namely, second order partial differential equations of hyperbolic type with constant coefficients. An interesting problem solved by him by using operational method was the determination of the propagation of disturbances in long transmission lines. He also used these methods to solve the partial differential equations occuring in the problems of electromagnetism and heat conduction. A detailed exposition of his ideas is given in the second volume of his celebrated work, Electromagnetic Theory [45].

The new concept of operational method is based on a functional transformation provided by Fourier and Laplace integrals. Many mathematicians have contributed to the development of the new concept in their attempt to interpret and justify the work of Heaviside. Indeed, Bromwich, Wagner and Carson all attempted to provide stable mathematical foundation for the Heaviside's operational calculus on the basis of contour integration.

Around 1916, Bromwich [4] solved the homogeneous system of n simultaneous ordinary differential equations of second order in n unknowns x_r by using the formula

$$x_r = \frac{1}{2\pi i} \int e^{\lambda t} X_r(\lambda) d\lambda \ ;$$

the path of integration, now known as Bromwich contour, is a closed path enclosing all the poles. After a gap of a decade, Bromwich [6] proved that the Heaviside function H is given by

$$H(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{pt}}{p} dp$$

where σ, t are positive.

Almost at the same time Wagner [126] in 1916, also dealt with the equation similar in nature to that of Bromwich, using different contour. However, his paper is more electrical oriented in its context.

In 1920's Carson [12] [13] who was more interested in the investigation on operational calculus formulated the operational method on the basis of infinite integrals of the Laplace type.

The present form of the method of Laplace transformation was given by the efforts of Doetsch [21], Vander Pol [122], [123] etc. who unified the earlier work into the current procedure of solutions.

Almost a decade later, Plesner [86] dealt with the operational calculus in the spectral theory of linear operators.

After 1940, the theory of integral transform has been enriched considerably by many mathematicians and we now sketch in brief some of these advancements in chronological order.

In his book [128] appeared in 1946, Widder discussed in detail the Laplace as well as Stieltjes transform given respectively by

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

and

$$F(s) = \int_0^\infty \frac{f(t)}{s+t}dt$$

where f is a single valued function of exponential order defined on (o, ∞) and having finite number of discontinuities. In particular, he proved that the Stieltjes transform is obtained by iteration of Laplace transform.

In 1949, Jaeger [50] published his monograph ''An Introduction to the Laplace Transformation with Engineering Applications'' , wherein he considered several applications of transformations to electric circuit theory, mechanical analogues to circuit problems, and a number of problems concerning transmission lines.

Having observed the limitations of operational calculus by the use of Laplace transforms from applications point of view, Mikusinski like Heaviside, initiated the study of operational calculus using algebra of functions where he considered the convolution of two functions as defining their product.

Aroung 1955, a variation of Laplace and Stieltjes transform was considered by Hirschman and Widder, which they termed the convolution transform given by

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\ \varphi(t)dt.$$

They proved real and complex inversion formulae for this transform and discussed the asymptotic behaviour of the kernel $G(t)$ as $t \to \pm\infty$. Also they showed the conversion of convolution transform to Laplace and Stieltjes transform by the suitable change of variable x in the kernel $G(x)$.

The study of the convolution transform was continued by Tanno who in a series of papers [111], [112], [113] considered

the following convolution transform

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\ e^{ct}\ d\alpha(t)$$

and proved its inversion formula. He also generalized several results obtained by Hirschman and Widder referred to above.

In 1962, another variation of Laplace transform known as the Mellin transform given by

$$T\{ f(x) \} = F(s) = \int_{0}^{\infty} f(x)\ x^{s-1}\ dx$$

was used by Lemon [67] to find the solution of Laplace equation in polar coordinates.

While studying Laplace transform, Dunn [24] replaced the infinite range (o, ∞) of integration to finite one (o, T) in order to get

$$F(s,t) = \int_{0}^{T} f(t)\ e^{-st}\ dt$$

He proved its inversion formula and showed that the class of functions operated by this transform is bigger than that of the Laplace transform ; for instance, it includes functions which are not necessarily of exponential order. Further he applied this transform to obtain the deflection of a taut string caused by a distributed load.

A good account of all these transformations, namely Fourier, Mellin, Convolution, Hankel is given in [104]; however, we

would also like to cite the names of Rooney [92], [93] and Angelina Byrne and Love [7], [70] and [71] for their contributions in fractional integration, applications of fractional integration to Mellin and Hankel transforms and generalized Stieltjes transform respectively. We may also refer to work of Titchmarsh [120], Tranter [121] and Watson [127].

Importance : Before we pass on to the historical development of generalized integral transformations, let us have a general review of the importance of the study of various types of integral transforms. Indeed, the method of integral transformation offers a powerful technique for the various areas in the field of applied mathematics. The class of functions that can be treated is extensive and includes those involved in many physical problems ; for instance, differential equations mentioned in the above paragraphs. In contrast to classical method, where one needs seperately the initial and boundary conditions for a specific solution, these conditions are automatically incorporated in the Laplace transform solution. The use of an integral transform often reduces a partial differential equation in n independent variables to (n-1) variables ; and successive operations of this type somewhat simplifies the problem to the solution of an ordinary differential equation or to the solution of an ordinary algebraic equation.

Although the Laplace transform, the use of which is widely known to physicists and engineers, is particularly suitable for problems governed by ordinary differential equations and problems in heat conduction , other integral transforms mentioned above, are also found to be useful in the solution of the boundary value problems of mathematical physics. There appears to be a good scope for the extension of this method by using different other kernels as well.

4. <u>GENERALIZED INTEGRAL TRANSFORMS</u> :

Having discussed a brief historical account of the theory of integral transforms and the theory of generalized functions in the last two sections, we now take up a brief development of their confluence, namely ''Generalized Integral Transforms''. There are several extensions of integral transforms to generalized functions. We consider them one by one in chronological order.

It was Schwartz [96] who in 1948, first considered the Fourier transform of generalized functions. Indeed, he proved the invariance of the space S (cf. Definition 1.4.2) and S^* (the space of spherical or tempered distributions) under the Fourier transform. Three years later in 1951, Schwartz [99] showed that an arbitrary distribution on euclidean space does not have a Fourier transform, whereas

on the otherside a periodic distribution always has a Fourier series. Following the lines of Bochner [3], he [100] also considered the Laplace transforms of generalized functions.

Around 1953 Gelfand and Shilov [35] extended the results of Schwartz on Fourier transforms. More specifically, they introduced the space Φ of infinitely differentiable functions defined on $\mathbb{R}^n$ (i.e. test functions), topologized suitably, such that the members φ of Φ behave at infinity in such a way that the Fourier transform, namely,

$$(*) \qquad \tilde{\varphi}(s) = \int_{\mathbb{R}^n} \exp[-2\pi i(s_1x_1+\ldots+s_nx_n)]\ \varphi(x)dx,$$

is defined for all φ where $s = (s_1,..,s_n)$, $x = (x_1,...,x_n)$ $\mathbb{R}^n$. Further, they defined the Fourier transform of a generalized function $T \in \Phi^*$, as a generalized function on $\tilde{\Phi}$ (collection of all $\tilde{\varphi}$'s defined by (*)) such that

$$\tilde{T}(\tilde{\varphi}) = T(\underline{\varphi})$$

where $\underline{\varphi}(x) = \varphi(-x)$. They also considered similar results for distributions generated by Lebesgue integrable function defined on compact sets.

A year later in [47], Horváth introduced the Hilbert transform of distributions. He also extended this notation to vector-valued distributions ; indeed, for the space D_{L^p} $(1 \leq p \leq \infty)$ consisting of functions φ which along with

all its derivatives belong to $L^p(\mathbb{R}^n)$; and its dual $D^*_{L^p} \equiv D_{L^q} (\frac{1}{p} + \frac{1}{q} = 1)$, he defined the Hilbert transform of T in $D^*_{L^p}$ as $K(T) = T * K$ where $K = \text{v.p.} \frac{k(\sigma)}{r^n}$; $r = |x|$, $\sigma = \frac{x}{r}$ and $k(\sigma)$ is a function defined on the unit sphere S in $\mathbb{R}^{n-1}$ satisfying the condition $\int_S k(\sigma)d\sigma = 0$, and $k(\sigma) \in L^s(S)$, $s > 1$.

In the late fifties, Silva [102] extended the results of Schwartz to define the Fourier transform of a very general kind of distributions.Indeed, Silva considered the Fourier transform of members of Z^* , known as the ultradistributions where Z is the space of complex analytic functions, and constructed a space of ultra-distributions which remains invariant under the Fourier transform introduced by him (note that the Fourier transform of a Schwartz distribution is, in general, an element of Z^*).

In early sixties, whereas Ishihara [49] considered the Laplace transform $\mathcal{L}(T)$ of a distribution T as a mapping $\xi \to \mathcal{F} \exp(-\xi y)\, T$ from $\mathbb{R}^n$ to the space of generalized Fourier transform of Gelfand and Shilov, Jones [52] obtained properties of Hilbert transforms for classes of generalized functions defined in terms of derivatives of functions in the L_p spaces, showing some applications of his results in electrical signal analysis.

Aroung 1966, Benedetto [2] introduced the Laplace transform of a distribution T in D^* with supp $T \subset [0, \infty]$, as a mapping from the set of complex numbers s with $Re(s) > \sigma_0$ for some σ_0, into the complex plane, given by

$$\mathcal{L}(T)(s) = \langle \exp(-\sigma_1 t)T \ , \ \exp\{-(s-\sigma_1)\} \ t \rangle ,$$

where $Re s > \sigma_1 > \sigma_0$ such that $\exp(-\sigma_0 t)T$ is a tempered distribution. He also derived integral expressions for $\mathcal{L}(T)(s)$ and discussed uniform convergence properties.

A great deal of work in the direction of Laplace, Mellin, Hankel, I, convolution, and Weierstrass transforms of distributions is due to Zemanian and his group, contained in a series of papers [61], [84], [132], [134] and [135]. According to Zemanian [132], the space $L_{a,b}$ consists of infinitely differentiable complex valued functions φ on $\mathbb{R}^n$ for which

$$(*) \qquad \max_{0 \leq k \leq p} \ \sup_t |k_{a,b}(t) \ D_t^k \ \varphi(t)| < \infty$$

where $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$, $a_r < b_r$, $r = 1, 2, \dots, n$;

$$k_{a,b}(t) = \pi \ k_{a_r, b_r} \ (t_r)$$

and

$$k_{a_r, b_r}(t_r) = \exp(a_r t_r) \ , \ t_r > 1$$

$$= \exp(b_r t_r) \ , \ t_r < -1.$$

Zemanian topologized the space $L_{a,b}$ with the help of seminorms, defined by the expressions (*). Assuming $e^{-st} \in L_{a,b}$, he defined the Laplace transform of $f \in L^*_{a,b}$ as follows

$$(Lf)(s) = < f, e^{-st} > .$$

Restriction of space prevents us from reproducing even certain quite many results of Zemanian and we refer to his monograph [136] for further detail , where one finds a very good account of all the transforms referred to above.

In 1972, Milton [78] generalized the results of Zemanian related to Fourier and Laplace transforms of a distribution by enlarging the class of distributions considered by Zemanian. He also proved some results on the asymptotic behaviour of these transforms, which are termed as Abelian theorems. Two years later in 1974, Milton [79] extended his own results in [78] . He introduced Fourier transforms of odd and even tempered distributions and proved that the intersection of the dual spaces of all even and odd functions in S (cf. Definition 1.4.2) is the dual of S.

In 1974, Lee, a student of Zemanian, attacked the problem concerning the extension of Hankel transform of distributions considered by Zemanian, by generalizing the space H_μ of Zemanian which consists of infinitely differentiable, complex valued functions φ defined on (o, ∞) such that

$$\gamma^{\mu}_{m,k}(\varphi) = \sup_{0<x<\infty} |x^m(x^{-1}D_x)^k\{x^{-(\frac{2\mu+1}{2})}\varphi(x)| < \infty,$$

to the three spaces $H_{\mu,\alpha}$, H_{μ}^{β} and $H_{\mu,\alpha}^{\beta}$ which he called the spaces of type H_μ. Besides discussing several properties of these spaces, he proved that the conventional Hankel transform defined by

$$\Phi(y) \equiv (h_\mu \varphi)(x) = \int_0^\infty \varphi(x) \sqrt{xy}\ J_\mu(xy)dx$$

is a continuous linear mapping from one space of the type H_μ into another space of the type H_μ for $\mu \geq -\frac{1}{2}$. He further defined Hankel transforms of generalized functions belonging to the dual of spaces of type H_μ and proved several results on the Hankel transforms of generalized functions. He showed the importance of his space by giving an application of these for solving a Dirichlet problem in cylindrical coordinates, the boundary conditions being given in terms of generalized functions and discussed the relationship of these spaces with certain classes of entire functions. Apart from these, Lee [65], [66] also extended the Hankel transforms of distributions introduced by A.L.Schwartz [94] as follows

$$[h_\mu \varphi(x)]y = \int_0^\infty m'(x)\ j_v(xy)\ \varphi(x)dx$$

where $m'(x) = [2^v \Gamma(v+1)]^{-1}\ x^{2v+1}$, $j_v(x) = \dfrac{2^v \Gamma(v+1)\ J_v(x)}{x^v}$.

He also showed the connection between the two notions.

In the year 1975, McBride [74] developed a theory of fractional integration for certain classes of generalized functions and introduced the space F_p and $F_{p,\mu}$ for $1 \leq p < \infty$ and $\mu \in \mathbb{C}$ as follows

$$F_p = \{\varphi : \varphi \in C^\infty \text{ and } \gamma_k^p(\varphi) \equiv |x^k \frac{d^k\varphi}{dx^k}|_p < \infty, k = 0, 1, \ldots \}$$

$$F_{p,\mu} = \{ \varphi : x^{-\mu} \varphi(x) \in F_p \}$$

where $|\Psi|_p = (\int_0^\infty |\Psi(x)|^p)^{1/p}$.

He used these spaces in [75] to define Hankel transforms H_ν of order $\nu (\nu \in \mathbb{C})$ for members of the space $F^*_{p,\mu}$ where $F^*_{p,\mu}$ is the dual of $F_{p,\mu}$ corresponding to the topology generated by the multinorm (separating collection of seminorms)

$$\{ \gamma_k^{p,\mu} : k \geq 0 \} , \quad \gamma_k^{p,\mu} (\varphi) = \gamma_k^p (x^{-\mu} \varphi).$$

He further discussed various restrictions or modifications of Hankel transform H_ν which maps $F^*_{p,\mu}$ to different spaces.

Almost simultaneously Dube and Pandey [22] introduced the Schwartz's Hankel transform of generalized functions and proved its inversion formula. Using their notion of generalized Hankel transform, they solved a Dirichlet problem in cylindrical coordinates. Extending the Gelfand and Shilov technique, Pandey [81] defined the Hilbert transform of a tempered distribution f on $\mathbb{R}^n$ and proved its inversion formula. He also modified and corrected the results of Gelfand and Shilov [37] on Hilbert

transforms. A year later, Pandey in collaboration with Chaudhary developed a theory for the Hilbert transform of generalized functions and applied their results for proving the existence and uniqueness of solutions to a Dirichlet boundary value problem. In a recent paper [82] Pandey has shown the subspace H(D) of $C^{\infty}(\mathbb{R})$, consisting of the Hilbert transforms of elements of D(cf. Definition 1.4.2) and equipped with an appropriate topology is homeomorphic to D under classical Hilbert transform. He has also defined Hilbert transform of $f \in D^{*}$ in terms of members of $H^{*}(D)$ as follows

$$\langle Hf, \varphi \rangle = \langle f, -H\varphi \rangle, \text{ for all } \varphi \in H(D).$$

Besides giving inversion formula for the above Hilbert transform, he has also discussed applications of his results in solving some singular integral equations.

No doubt we have tried to uptodate the recent advances in the theory of generalized integral transforms. However, our bibliography would remain incomplete if we donot mention the important contributions due to Ehrenpreis [25], Erdelyi [26], Lighthill [68], Liverman [69] and several references given there in.

Finally, motivated by the work of Gelfand and Shilov., Lee and Zemanian, the author of the present thesis got interested in combining the Laplace [132] and Hankel [136] transformations in order to define Laplace-Hankel transform for two variables

nd finally succeeded in generalizing almost all the results of Lee [64]. The present thesis includes this work as well. However, there are still many problems open in this direction, for instance establishing the relationship the spaces of type LH with classes of entire functions, Laplace-Hankel transform of arbitrary order etc.

As the wheel of time rolls by, we will keep running the hope to receive more and more progress that is expected pour into the arena of our wisdom!.

CHAPTER -3

SPACES OF TYPE LH

CONTENTS

1. INTRODUCTION :

In this chapter we consider several different LH-spaces consisting of exponentially decreasing infinitely differentiable functions defined on the first and second quadrants of the plane. We make a systematic study of the topological structures of these spaces and investigate their interrelationship. In the last part of this chapter, we also pay attention to the inductive limits obtained from these various spaces.

2. VARIOUS LH-SPACES: To begin our study on several LH-spaces along with their natural Hausdorff locally convex topologies, let us denote by Ω_1 and Ω_2 the open sets $(\mathbb{R}_+)\times(\mathbb{R}_+)$ [first quadrant] and $(\mathbb{R}_-)\times(\mathbb{R}_+)$ [second quadrant] respectively. Further the classes of infinitely differentiable functions defined on Ω_1 and Ω_2 are respectively symbolized by $E(\Omega_1)$ and $E(\Omega_2)$. We use the notation Δ for the differential operator $x^{-1}D_x$. Unless otherwise specified, we choose a,μ,α,β from $\mathbb{R}$ with $\alpha,\beta \geq 0$ and the letters k,q,l mean $0,1,2\ldots$. Also the notation 0^0 whereever it occurs in the sequel, means the number 1. We are now prepared to introduce

2.1 THE SPACE $LH_{a,\mu,\alpha}$: It is defined as

$$LH_{a,\mu,\alpha} = \{\varphi : \varphi \in E(\Omega_1);$$

$$\gamma^{\mu}_{a,k,q,l}(\varphi) = \sup_{\Omega_1} |x^k e^{at} D_t^l \Delta^q (x^{-(\frac{2\mu+1}{2})}\varphi(t,x))|$$

$$\leq C_{ql}\, A^k\, k^{k\alpha}\},$$

where the constants A and C_{ql} depend on the function φ.

2.2 <u>THE SPACE</u> $LH^{\beta}_{a,\mu}$: It is given by

$$LH^{\beta}_{a,\mu} = \{ \varphi : \varphi \in E(\Omega_1);$$

$$\gamma^{\mu}_{a,k,q,l}(\varphi) = \sup_{\Omega_1} | x^k e^{at} D^l_t \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi(t,x)\} |$$
$$\leq C_{kl} B^q q^{q\beta} \} .$$

The constants B and C_{kl} depend on the function φ.

2.3 <u>THE SPACE</u> $LH^{\beta}_{a,\mu,\alpha}$: This space is defined as follows :

$$LH^{\beta}_{a,\mu,\alpha} = \{ \varphi : \varphi \in E(\Omega_1);$$

$$\gamma^{\mu}_{a,k,q,l}(\varphi) = \sup_{\Omega_1} | x^k e^{at} D^l_t \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi(t,x)\} |$$
$$\leq C_l A^k B^q k^{k\alpha} q^{qB} \} ,$$

where the constants A, B and C_l depend on the function φ.

Analogous to 2.1, 2.2, 2.3 we can define the spaces of functions which have domain Ω_2. However, for the sake of notational convenience, we mention them in

2.4 <u>THE SPACE</u> $\overset{\vee}{LH}_{a,\mu,\alpha}$: It is given by

$$\overset{\vee}{LH}_{a,\mu,\alpha} = \{ \varphi : \overset{\vee}{\varphi}(t,x) = \varphi(-t,x) \in LH_{a,\mu,\alpha} ;$$

$$\rho^{\mu}_{a,k,q,l}(\varphi) = \sup_{\Omega_2} | x^k e^{-at} D^l_t \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi(t,x)\} |$$
$$\leq C_{ql} A^k k^{k\alpha} \} .$$

The constants A and C_{ql} depend on the function φ.

2.5 THE SPACE $\overset{v}{LH}{}^{\beta}_{a,\mu}$: We define it as

$$\overset{v}{LH}{}^{\beta}_{a,\mu} = \{ \varphi : \overset{v}{\varphi}(t,x) = \varphi(-t,x) \in LH^{\beta}_{a,\mu} ;$$

$$\rho^{\mu}_{a,k,q,l}(\varphi) \equiv \sup_{\Omega_2} | x^k e^{-at} D^l_t \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi(t,x) \} |$$

$$\leq C_{kl} B^q q^{q\beta} \} ,$$

where the constants B and C_{ql} depend on the function φ.

2.6 THE SPACE $\overset{v}{LH}{}^{\beta}_{a,\mu,\alpha}$: This is formed by combining the conditions of 2.4 and 2.5 as given in

$$\overset{v}{LH}{}^{\beta}_{a,\mu,\alpha} = \{ \varphi : \overset{v}{\varphi}(t,x) \equiv \varphi(-t,x) \in LH^{\beta}_{a,\mu,\alpha} ;$$

$$\rho^{\mu}_{a,k,q,l}(\varphi) \equiv \sup_{\Omega_2} | x^k e^{-at} D^l_t \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi(t,x) \} |$$

$$\leq C_l A^k B^q k^{k\alpha} q^{q\beta} \} .$$

The constants A, B and C_l depend on the function φ.

Following the order of the above spaces, we now introduce subspaces of each of the above spaces used in defining the inductive limits in the last section of this chapter.

2.7 THE SPACE $LH_{a,\mu,\alpha,m}$: For given m > 0, this is the subspace of the space 2.1, given by

$$LH_{a,\mu,\alpha,m} = \{ \varphi : \varphi \in LH_{a,\mu,\alpha} ;$$

$$\gamma^{\mu}_{a,k,q,l}(\varphi) \equiv \sup_{\Omega_1} | x^k e^{at} D^l_t \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi(t,x) \} |$$

$$\leq C_{ql\delta}(m+\delta)^k k^{k\alpha}\}$$

where δ is any number > 0.

2.8 <u>THE SPACE</u> $LH^{\beta,n}_{a,\mu}$: For given $n > 0$, we set

$LH^{\beta,n}_{a,\mu} = \{\varphi : \varphi \in LH^{\beta}_{a,\mu}$;

$$\gamma^{\mu}_{a,k,q,l}(\varphi) \equiv \sup_{\Omega_1} | x^k e^{at} D^l_t \Delta^q \{x^{-(\frac{2\mu+1}{2})} \varphi(t,x)\}|$$
$$\leq C_{kl\eta}(n+\eta)^q q^{q\beta}\},$$

where η is any number greater than 0.

2.9 <u>THE SPACE</u> $LH^{\beta,n}_{a,\mu,\alpha,m}$: We define this space as

$LH^{\beta,n}_{a,\mu,\alpha,m} = \{\varphi : \varphi \in LH^{\beta}_{a,\mu,\alpha,m}$;

$$\gamma^{\mu}_{a,k,q,l}(\varphi) \equiv \sup_{\Omega_1} |x^k e^{at} D^l_t \Delta^q \{x^{-(\frac{2\mu+1}{2})} \varphi(t,x)\}|$$
$$\leq C_{\delta\eta}(m+\delta)^k (n+\eta)^q k^{k\alpha} q^{q\beta}\},$$

for given $m,n > 0$ and for any $\delta,\eta > 0$.

2.10 <u>THE SPACE</u> $\check{L}H_{a,\mu,\alpha,m}$: This is defined as

$\check{L}H_{a,\mu,\alpha,m} = \{\varphi : \check{\varphi}(t,x) \equiv \varphi(-t,x) \in LH_{a,\mu,\alpha,m}$;

$$\rho^{\mu}_{a,k,q,l}(\varphi) \equiv \sup_{\Omega_2} |x^k e^{-at} D^l_t \Delta^q \{x^{-(\frac{2\mu+1}{2})} \varphi(t,x)\}|$$

$$\leq C_{ql\delta}(m+\delta)^k k^{k\alpha}\},$$

where m is a given positive number and δ is any number greater than 0.

2.11 <u>THE SPACE</u> $\overset{v}{LH}{}^{\beta,n}_{a,\mu}$: For given $n > 0$, this is the subspace of the space 2.5, given by

$$\overset{v}{LH}{}^{\beta,n}_{a,\mu} = \{ \varphi : \overset{v}{\varphi}(t,x) \equiv \varphi(-t,x) \in LH^{\beta,n}_{a,\mu} ;$$

$$\rho^{\mu}_{a,k,q,l}(\varphi) = \sup_{\Omega_2} | x^k e^{-at} D^l_t \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi(t,x) \} |$$

$$\leq C_{kl\eta} \; (n+\eta)^q \; q^{q\beta} \} ,$$

where η is any number greater than 0.

2.12 <u>THE SPACE</u> $\overset{v}{LH}{}^{\beta,n}_{a,\mu,\alpha,m}$: We define this space as

$$\overset{v}{LH}{}^{\beta,n}_{a,\mu,\alpha,m} = \{ \varphi : \overset{v}{\varphi}(t,x) = \varphi(-t,x) \in LH^{\beta,n}_{a,\mu,\alpha,m} ;$$

$$\rho^{\mu}_{a,k,q,l}(\varphi) \equiv \sup_{\Omega_2} | x^k e^{-at} D^l_t \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi(t,x) \} |$$

$$\leq C_{\delta\eta} \; (m+\delta)^k \; (n+\eta)^q \; k^{k\alpha} \; q^{q\beta} \} ,$$

for given $m, n > 0$ and any $\delta, \eta > 0$.

Lastly we introduce

2.13 <u>THE SPACE</u> $LB_{a,\mu,m}$: For given $m > 0$, this is the space of smooth functions vanishing on $\mathbb{R}_+ \times (m,\infty)$, namely

$$LB_{a,\mu,m} = \{ \varphi : \varphi = 0 \text{ for } (\mathbb{R}_+) \times (m,\infty) ;$$

$$\gamma^{\mu}_{a,q,1}(\varphi) \equiv \sup_{\Omega_1} | e^{at} D_t^1 \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi(t,x) \} |$$

$$< \infty \ \} .$$

Unless specified otherwise, the spaces introduced in (2.1) through (2.13) will henceforth be considered equipped with their natural Hausdorff locally convex topologies to be denoted respectively by $T_{a,\mu,\alpha}$, $T^{\beta}_{a,\mu}$, $T^{\beta}_{a,\mu,\alpha}$, $\overset{v}{T}_{a,\mu,\alpha}$, $\overset{v}{T}{}^{\beta}_{a,\mu}$, $\overset{v}{T}{}^{\beta}_{a,\mu,\alpha}$, $T_{a,\mu,\alpha,m}$, $T^{\beta,n}_{a,\mu}$, $T^{\beta,n}_{a,\mu,\alpha,m}$, $\overset{v}{T}_{a,\mu,\alpha,m}$, $\overset{v}{T}{}^{\beta,n}_{a,\mu}$, $\overset{v}{T}{}^{\beta,n}_{a,\mu,\alpha,m}$ and $T_{a,\mu,m}$. These topologies are respectively generated by the total families of seminorms $\{\gamma^{\mu}_{a,k,q,1}\}$, $\{\gamma^{\mu}_{a,k,q,1}\}$, $\{\gamma^{\mu}_{a,k,q,1}\}$, $\{\rho^{\mu}_{a,k,q,1}\}$, $\{\rho^{\mu}_{a,k,q,1}\}$, $\{\rho^{\mu}_{a,k,q,1}\}$, $\{\gamma^{\mu}_{a,k,q,1}\}$, $\{\gamma^{\mu}_{a,k,q,1}\}$, $\{\gamma^{\mu}_{a,k,q,1}\}$, $\{\rho^{\mu}_{a,k,q,1}\}$, $\{\rho^{\mu}_{a,k,q,1}\}$, $\{\rho^{\mu}_{a,k,q,1}\}$ and $\{\gamma^{\mu}_{a,q,1}\}$.

3. THE SPACES DEFINED ON Ω_1:

This section includes results exhibiting relationships amongst various LH-spaces defined on Ω_1 in the preceding section. However, we begin our discussion with the following useful

THEOREM 3.1 : $(LH_{a,\mu,\alpha} ; T_{a,\mu,\alpha})$ is a Frechet space.

PROOF : As the family $D_{a,\mu,\alpha}$ of seminorms generating $T_{a,\mu,\alpha}$ is countable, it suffices to prove the completeness of the space $(LH_{a,\mu,\alpha} ; T_{a,\mu,\alpha})$.

Let us therefore consider a Cauchy sequence $\{\varphi_n\}$ in $LH_{a,\mu,\alpha}$.

Hence for any given $\varepsilon > 0$, there exists an $N \equiv N_{kql}$ such that for $m,n \geq N$

$$(3.2) \quad \gamma^{\mu}_{a,k,q,l}(\varphi_m-\varphi_n) = \sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left[x^{-(\frac{2\mu+1}{2})} \cdot \{\varphi_m(t,x)-\varphi_n(t,x)\} \right] \right| < \varepsilon$$

In particular, for $k,q,l = 0$

$$(3.3) \quad \sup_{\Omega_1} | e^{at} x^{-(\frac{2\mu+1}{2})} \{ \varphi_m(t,x) - \varphi_n(t,x)\} | < \varepsilon, \; m,n \geq N.$$

Consequently, for fixed (t,x) in Ω_1, $\{\varphi_m(t,x)\}$ is a numerical Cauchy sequence. Let $\varphi(t,x)$ be the pointwise limit of $\{\varphi_m(t,x)\}$,

Using (3.3), we can easily deduce that $\{\varphi_m\}$ converges to φ uniformly on Ω_1. Thus φ is continuous. Moreover repeated use of (3.2) for different values of q and l and the use of Proposition 1.7.3 yield that φ is smooth, i.e. $\varphi \in E(\Omega_1)$.

Further from (3.2), we get

$$\gamma^{\mu}_{a,k,q,l}(\varphi_m) \leq \gamma^{\mu}_{a,k,q,l}(\varphi_N) + \varepsilon , \quad \forall\, m \geq N$$

$$\leq C_{kl}\, A^k k^{k\alpha} + \varepsilon, \quad \forall\, m \geq N$$

Letting $m \to \infty$ and observing that ε is arbitrary, we get

$$\gamma^{\mu}_{a,k,q,l}(\varphi) = \sup_{\Omega_1} |x^k e^{at} D_t^l \Delta^q \{x^{-(\frac{2\mu+1}{2})} \varphi\}| \leq C_{kl} A^k k^{k\alpha}$$

Hence $\varphi \in LH_{a,\mu,\alpha}$ and it is the $T_{a,\mu,\alpha}$ limit of φ_m by (3.2) again.#

In order to justify our study of LH-spaces, i.e. the nontriviality of these spaces, we show that $D(\Omega_1)$ is a proper subspace of each of these spaces. The proof of this result makes use of the basic

LEMMA 3.4: For $\varphi \in LH_{a,\mu,\alpha}$,

$$(3.5)\quad \Delta^q \{x^{-(\frac{2\mu+1}{2})} \varphi\} = \sum_{j=o}^{q} a_j\, x^{-(\frac{2\mu+1}{2})-q-j} D_x^{q-j} \varphi ,$$

where a_j's are some constants.

PROOF: For $q = 1$, the result is direct differentiation of product of two functions.
For using induction, let (3.5) be true for $q = k - 1$.

Consider

$$\Delta^k \{x^{-(\frac{2\mu+1}{2})} \varphi\} = \Delta[\Delta^{k-1}\{x^{-(\frac{2\mu+1}{2})} \varphi\}]$$

$$= \Delta\Big[\sum_{j=o}^{k-1} a_j\, x^{-(\frac{2\mu+1}{2})-(k-1)-j} D_x^{k-1-j} \varphi\Big]$$

$$= \sum_{j=o}^{k} a'_j\, x^{-(\frac{2\mu+1}{2})-k-j} D_x^{k-j} \varphi \;.\;\#$$

<u>THEOREM</u> 3.6: The space $D(\Omega_1)$ is a subspace of $LH_{a,\mu,\alpha}$, such that the injection map from $D(\Omega_1)$ to $LH_{a,\mu,\alpha}$ is continuous, i.e. $T_{a,\mu,\alpha} \mid D(\Omega_1) \subset T_{\Omega_1}$.

<u>PROOF</u>: For $\varphi(t,x) \in D(\Omega_1)$, set

$$L = \sup\{ x : (t,x) \in \operatorname{supp}(\varphi)\};$$

and

$$C_{ql} = \sup_{\Omega_1} | e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi \}| .$$

Then

$$(3.7) \qquad \begin{aligned} \sup_{\Omega_1} | x^k e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi\}| &\le C_{ql} L^k \\ &\le C_{ql} (\frac{L}{Ak^{\alpha}})^k A^k k^{k\alpha} \end{aligned}$$

Since $(\frac{L}{Ak^{\alpha}}) \le 1$ if and only if $k \ge (\frac{L}{A})^{\frac{1}{\alpha}}$, define $k_o = [(\frac{L}{A})^{1/\alpha}]+1$, where $[x]$ denotes the Gaussian symbol, that is the greatest integer not exceeding x. Therefore for $k > k_o$, we have from (3.7)

$$(3.8) \quad \sup_{\Omega_1} | x^k e^{at} D_t^l \Delta^q \{x^{-(\frac{2\mu+1}{2})} \varphi\}| \le C_{ql} A^k k^{k\alpha}.$$

If $k \le k_o$, let us write

$$C = \max \{ \frac{L}{A}, (\frac{L}{A2^{\alpha}})^2, \dots, (\frac{L}{Ak_o^{\alpha}})^{k_o} \}.$$

Then, again from (3.7)

$$(3.9) \quad \sup_{\Omega_1} |x^k e^{at} D_t^l \Delta^q \{x^{-(\frac{2\mu+1}{2})}\varphi\}| \leq C_{ql} \, CA^k \, k^{k\alpha}$$

Hence the inequalities (3.8) and (3.9) together yield

$$\gamma^{\mu}_{a,k,q,l}(\varphi) \leq C'_{ql} \, A^k k^{k\alpha}, \qquad \forall \, k \geq 0,$$

implying $\varphi \in LH_{a,\mu,\alpha}$, consequently $D(\Omega_1) \subset LH_{a,\mu,\alpha}$.

To prove the continuity of the injection map, consider a sequence $\{\varphi_n\}$ in $D(\Omega_1)$, converging to zero. Then by Lemma 3.4,

$$\gamma^{\mu}_{a,k,q,l}(\varphi_n) = \sup_{\Omega_1} |x^k e^{at} D_t^l \sum_{j=0}^{q} a_j \, x^{-(\frac{2\mu+1}{2})-q-j} D_x^{q-j} \varphi_n|$$

$$\leq \sum_{j=0}^{q} \sup_{\Omega_1} |e^{at} a_j \, x^{k-(\frac{2\mu+1}{2})-q-j} D_t^l D_x^{q-j} \varphi_n|$$

$$(3.10) \qquad \leq \sum_{j=0}^{q} \sup_{\Omega_1} |\{e^{at} a_j \, x^{k-(\frac{2\mu+1}{2})-q-j}\}\{D^{l+q-j} \varphi_n\}|$$

Now, using Theorem 1.2.5, we find a compact subset K_m (say) of Ω_1 such that $\varphi_n \in D(K_m)$ for each $n \geq 1$ and $\varphi_n \to 0$ in $D(K_m)$. If

$$M_j = \sup_{K_m} |e^{at} a_j x^{k-(\frac{2\mu+1}{2})-q-j}| ;$$

then from (3.10)

$$\gamma^{\mu}_{a,k,q,l}(\varphi_n) \leq \sum_{j=0}^{q} M_j \sup_{K_m} |D^{l+q-j} \varphi_n|$$

$\to 0$ as $n \to \infty$.

Hence $\varphi_n \to 0$ in $LH_{a,\mu,\alpha}$ and therefore $T_{a,\mu,\alpha}\,|\,D(\Omega_1) \subset T_{\Omega_1}$.

#

The following example illustrates the proper containment of $D(\Omega_1)$ in $LH_{a,\mu,\alpha}$.

<u>Example 3.11</u>: Consider a function $\varphi : \Omega_1 \to \mathbb{C}$ defined by

$$\varphi(t,x) = x^{\left(\frac{2\mu+1}{2}\right)} \exp\{-at - \frac{Bx^2}{2}\}, \qquad A^2B \geq 1.$$

We show that $\varphi \in LH_{a,\mu,\alpha}$. Indeed,

$$\sup_{\Omega_1} | x^k e^{at} D_t^l \Delta^q \{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi \} |$$

$$= \sup_{\Omega_1} | x^k (-a)^l (-B)^q \exp(-\tfrac{1}{2} Bx^2) |$$

$$\leq \sup_{\Omega_1} | x^k (a)^l \, (B)^q \exp(-\frac{x^2}{2A^2}) |$$

$$\leq \sup_{\Omega_1} \; C_1 B^q A^k \,|\, u^k \exp(-\tfrac{u^2}{2}) | \;, \; u = \tfrac{x}{A}$$

$$\leq C_1 B^q A^k \sup_{\Omega_1} | \{ u^2 \exp(-\tfrac{u^2}{2k}) \}^{\frac{k}{2}} |$$

$$(*) \qquad \leq C_1 B^q A^k k^{k/2}$$

$$\leq D_{ql} A^k k^{k/2} \;, \; D_{ql} = C_1 B^q.$$

Thus $\varphi \in LH_{a,\mu,\alpha}$.

It is clear that φ has no compact support and hence $\varphi \notin D(\Omega_1)$. This proves $D(\Omega_1) \subsetneq LH_{a,\mu,\alpha}$.

Coming to the other kind of LH-spaces on Ω_1, we have

THEOREM 3.12: $(LH^{\beta}_{a,\mu} ; T^{\beta}_{a,\mu})$ and $(LH^{\beta}_{a,\mu,\alpha} ; T^{\beta}_{a,\mu,\alpha})$ are Fréchet spaces.

PROOF: Analogous to the proof of Theorem 3.1 and so omitted.

For the nontriviality of these spaces, we offer

EXAMPLE 3.13 : Consider a function $\varphi : \Omega_1 \to \mathbb{C}$ defined by

$$\varphi(t,x) = x^{(\frac{2\mu+1}{2})} \exp\{ -at - \tfrac{1}{2} bx^2 \} , \; b \geq 1.$$

We show that $\varphi \in LH^{\beta}_{a,\mu}$. Indeed,

$$\sup_{\Omega_1} | x^k e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi \} |$$

$$= \sup_{\Omega_1} | x^k (-a)^l \Delta^q \{ \exp(-\tfrac{1}{2} bx^2) \} |$$

$$\leq C_{kl} b^q, \; C_{kl} = \sup_{x \in \mathbb{R}_+} | a^l x^k \exp(-\tfrac{1}{2} bx^2) |$$

$$\leq C_{kl} (\frac{b}{Bq^{\beta}})^q B^q q^{q\beta}.$$

Proceeding on lines similar to those in Theorem 3.6, we get

$$\sup_{\Omega_1} | x^k e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi \} |$$

$$\leq D_{kl} B^q q^{q\beta}$$

where $\quad D_{kl} = \max \{ C_{kl} \cdot C, C_{kl} \}$,

$$C = \max\left[\frac{b}{B}, \left(\frac{b}{B2^{\beta}}\right)^2, \ldots, \left(\frac{b}{Bq_o^{\beta}}\right)^{q_o}\right]$$ and

$q_o = \left[\left(\frac{b}{B}\right)^{\frac{1}{\beta}}\right] + 1$. Thus $\varphi \in LH^{\beta}_{a,\mu}$.

EXAMPLE 3.14: Let us recall the function φ of Example 3.11 and the inequality (*), which yields $\varphi \in LH^{\beta}_{a,\mu,\alpha}$.

Concerning the relationship among the spaces $LH_{a,\mu,\alpha}$, $LH^{\beta}_{a,\mu}$ and $LH^{\beta}_{a,\mu,\alpha}$ we have

THEOREM 3.15 : The space $LH^{\beta}_{a,\mu,\alpha}$ is contained in $LH_{a,\mu,\alpha} \cap LH^{\beta}_{a,\mu}$. Further the topology $T^{\beta}_{a,\mu,\alpha}$ of $LH^{\beta}_{a,\mu,\alpha}$ coincides with the topologies $T_{a,\mu,\alpha} \mid LH^{\beta}_{a,\mu,\alpha}$ and $T^{\beta}_{a,\mu} \mid LH^{\beta}_{a,\mu,\alpha}$.

PROOF : For $\varphi \in LH^{\beta}_{a,\mu,\alpha}$, we have

$$\gamma^{\mu}_{a,k,q,l}(\varphi) \leq C_1 A^k B^q k^{k\alpha} q^{q\beta}.$$

By adjusting the constants appropriately on the right hand side of this inequality, we easily infer $\varphi \in LH_{a,\mu,\alpha} \cap LH^{\beta}_{a,\mu}$.

The last part is an easy consequence of the definition of the topologies $T_{a,\mu,\alpha}$, $T^{\beta}_{a,\mu}$ and $T^{\beta}_{a,\mu,\alpha}$.

For proving the inclusion relation between two LH-spaces (Theorem 3.18), we need

LEMMA 3.16 : If $p \in \mathbb{N}$ and is even then

$$(*) \quad \Delta^j(x^p) = p(p-2)\ldots\{p-2(j-1)\}\, x^{p-2j};\ j = 1, 2, \ldots, \frac{p}{2}.$$

PROOF : We prove the above relation by induction. Clearly the relation (*) is true for $j = 1$. Let (*) be true for

$j = k-1$, so that

$$\Delta^{k-1}(x^p) = p(p-2)\ldots\{p-2(k-2)\}\, x^{p-2(k-1)}.$$

Consider

$$\Delta^k(x^p) = \Delta(\Delta^{k-1}x^p)$$

$$= p(p-2)\ldots\{p-2(k-1)\}\, x^{p-2k}.\ \#$$

LEMMA 3.17: For $\varphi \in LH_{a,\mu,\alpha}$, let $\theta = x^{-(\frac{2\mu+2p+1}{2})}\varphi$, where p is an even natural number. Then

$$(*)\qquad \Delta^q(x^p\theta) = \sum_{j=0}^{q} \binom{q}{j} \Delta^j x^p \Delta^{q-j}\theta$$

PROOF: Again we use induction to prove this result. For $q = 1$, let us note that

$$\Delta(x^p\theta) = \theta(\tfrac{1}{x}D_x)x^p + x^p(\tfrac{1}{x}D_x)\theta$$

$$= \theta\Delta x^p + x^p\Delta\theta$$

Hence, we need prove (*) for $q = k$, whenever it is true for $q = k-1$. Therefore

$$\Delta^k(x^p\theta) = \Delta(\Delta^{k-1}(x^p\theta))$$

$$= \sum_{j=0}^{k} \binom{k}{j} \Delta^j x^p . \Delta^{k-j}\theta .\#$$

We now prove

THEOREM 3.18: Let p be an even integer in $\mathbb{N}$. Then $LH_{a,\mu+p,\alpha} \subset LH_{a,\mu,\alpha}$, and also the induced topology on

$LH_{a,\mu+p,\alpha}$ is weaker than its original topology; i.e. $T_{a,\mu,\alpha} \mid LH_{a,\mu+p,\alpha} \subset T_{a,\mu+p,\alpha}$.

PROOF : The definition of the spaces $LH_{a,\mu,\alpha}$, $LH_{a,\mu+p,\alpha}$ and their topologies $T_{a,\mu,\alpha}$, $T_{a,\mu+p,\alpha}$ suggest that both the parts of the above theorem will hold good if we show the domination of a seminorm in $LH_{a,\mu,\alpha}$ by a finite combination of members of $D_{a,\mu+p,\alpha}$. Let us, therefore, take a seminorm $\gamma^{\mu}_{a,k,q,l}$ in $D_{a,\mu,\alpha}$ and consider for $\varphi \in LH_{a,\mu,\alpha}$. Then using Lemmas 3.16 and 3.17, we obtain

$$\gamma^{\mu}_{a,k,q,l}(\varphi) = \sup_{\Omega_1} | x^k e^{at} D_t^l \Delta^q \{ x^p . x^{-(\frac{2\mu+2p+1}{2})} \varphi \} |$$

$$= \sup_{\Omega_1} | x^k e^{at} D_t^l \sum_{j=0}^{q} \Delta^j x^p . \Delta^{q-j} \{ x^{-(\frac{2\mu+2p+1}{2})} \varphi \} | \text{ , Lemma 3.17}$$

$$= \sup_{\Omega_1} | \sum_{j=0}^{q} x^{k+p-2j} e^{at} D_t^l \binom{q}{j} [p(p-2)\ldots \{p-2(j-1)\}] \Delta^{q-j} \{ x^{-(\frac{2\mu+2p+1}{2})} \varphi \} | \text{, Lemma 3.16}$$

$$\leq \sum_{j=0}^{q} c_j \; \gamma^{\mu+p}_{a,k+p-2j,q-j,l}(\varphi),$$

where c_j's are some constants.

Since $\{ \gamma^{\mu+p}_{a,k+p-2j,q-j,l} \} \in D_{a,\mu+p,\alpha}$, the result follows.#

The following result useful for our next theorem, exhibits

the exponential behaviour of the members of $LH_{a,\mu,\alpha}$.

THEOREM 3.19 : For $\varphi \in LH_{a,\mu,\alpha}$ and $\alpha > 0$, the following inequality holds :

$$|D_t^l \Delta^q \; x^{-(\frac{2\mu+1}{2})} \varphi\}| \leq D_{ql} \exp\{-bx^{1/\alpha} - at\},$$

where $b = \{\alpha/(eA^{1/\alpha})\}$

PROOF : Let $\varphi \in LH_{a,\mu,\alpha}$, where $\alpha > 0$. Then

$$|e^{at} x^k D_t^l \Delta^q \{x^{-(\frac{2\mu+1}{2})} \varphi\}| \leq C_{ql} A^k k^{k\alpha}, \; \forall \; (t,x) \in \Omega_1.$$

Dividing both the sides by $x^k e^{at}$ and taking the infimum over k on the right hand side we get

$$|D_t^l \Delta^q \{x^{-(\frac{2\mu+1}{2})} \varphi\}| \leq \{C_{ql} \; e^{-at}\} \inf_k \frac{k^{k\alpha}}{T^k}, \; T = \frac{x}{A}$$

$$(*) \qquad \leq C_{ql} \; e^{-at} \inf_k f(k), \; f(k) = \frac{k^{k\alpha}}{T^k}.$$

Since k assumes only integral values, we find min $\{f(u) : u \geq 0\}$.

Using the elementary calculus method for computation of maxima and minima, we find $\min_{u \geq 0} f(u)$ is attained at $u_o = (\frac{1}{e}) T^{1/\alpha}$. Thus

$$(**) \qquad \min_{u \geq 0} f(u) = e^{(-\alpha/e) T^{1/\alpha}}$$

For obtaining $\inf_k f(k)$, write $k_o = [u_o] + 1$. Let $0 \leq h \leq 1$, be such that $k_o = u_o + h$. Then

$$f(k_o) = \frac{(u_o+h)^{(u_o+h)\alpha}}{T^{(u_o+h)}} = \frac{u_o^{(u_o+h)\alpha}}{T^{u_o+h}} \left[1 + \frac{h}{u_o}\right]^{(u_o+h)\alpha}$$

$$= f(u_o) \frac{(\frac{1}{e} T^{\frac{1}{\alpha}})^{h\alpha}}{T^h} (1 + \frac{h}{u_o})^{u_o\alpha} (1 + \frac{h}{u_o})^{h\alpha}$$

$$\leq f(u_o) (1 + \frac{h}{u_o})^{h\alpha}$$

$$(+) \qquad \leq \exp(- \frac{\alpha}{e} T^{1/\alpha}) (1 + \frac{he}{T^{1/\alpha}})^{h\alpha}$$

We now consider two cases : (i) $T \geq 1$ and (ii) $T < 1$.

(i) For $T \geq 1$, $T^{-1/\alpha} \leq 1$ and so from (+),

$$(++) \qquad f(k_o) \leq A \exp(- \frac{\alpha}{e} T^{1/\alpha})$$

where $A = (1+he)^{h\alpha}$, a constant

(ii) For $T < 1$, observe that

$$\min_k f(k) \leq 1 \leq \exp(\frac{\alpha}{e}) \exp(- \frac{\alpha}{e} T^{1/\alpha}).$$

Thus

$$(+++) \qquad \min_k f(k) \leq B \exp(- \frac{\alpha}{e} T^{1/\alpha}),$$

$B = \exp(\alpha/e)$, a constant.

Hence from (++) and (+++), we get

$$(***) \qquad \min_k f(k) < C \exp(- \frac{\alpha}{e} T^{1/\alpha})$$

for all values of T.

Also, further from (*) and (***),

$$|D_+^l \Delta^q\{x^{-(\frac{2\mu+1}{2})}\varphi\}| \leq C_{ql}\cdot C\exp(-\frac{\alpha}{e}T^{1/\alpha} - at)$$

$$\leq D_{ql}\exp[-bx^{1/\alpha} - at],$$

where $b = \{\alpha/(eA^{1/\alpha})\}$. #

NOTE : Let us also point out the following inequalities obtained from (**) and (***):

(3.20) $$\exp(-\frac{\alpha}{e}T^{1/\alpha}) < \inf_k f(k) < C\exp(-\frac{\alpha}{e}T^{1/\alpha}).$$

Finally in this section, we consider two equivalent topologies on $LH_{a,\mu,\alpha,m}$, which are stronger than $T_{a,\mu,\alpha,m}$ as introduced in section 2. To begin with, let us introduce

$$M_p(t,x) = \exp[b(1-\frac{1}{p})x^{1/\alpha} + at]$$

where $b = \{\alpha/(em^{1/\alpha})\}$, $p = 2,3,\ldots$ Then we have

PROPOSITION 3.21: For $\varphi \in LH_{a,\mu,\alpha,m}$, one has

$$||\varphi||_p^\mu = \sup_{\substack{q\leq p\\ l\geq 0}}[\sup_{\Omega_1} M_p(t,x)\,|D_t^l \Delta^q\{x^{-(\frac{2\mu+1}{2})}\varphi\}|] < \infty$$

and $||\cdot||_p^\mu$ defines a norm on $LH_{a,\mu,\alpha,m}$.

PROOF : By Theorem 3.19, for $\varphi \in LH_{a,\mu,\alpha,m}$, we have

$$||\varphi||_p^\mu \leq \sup_{\substack{q\leq p\\ l\geq 0}}\ \sup_{\Omega_1}[D_{ql}\exp\{b(1-\frac{1}{p})x^{1/\alpha} + at\}.$$

$$.\exp\{-bx^{1/\alpha} - at\}]$$

$$= \sup_{\substack{q \leq p \\ l \geq 0}} \sup_{\Omega_1} D_{ql} \exp\left(-\frac{b}{p} x^{1/\alpha}\right)$$

$< \infty .$

Hence $||.||_p^{\mu}$ is well defined for each φ on $LH_{a,\mu,\alpha,m}$. Clearly it is a norm on this space.#

DEFINITION 3.22: Let D_{μ} denote the family of norms $\{||\varphi||_p^{\mu}$ $p = 2,3,\ldots\}$ as defined in Proposition 3.21 and T_{μ} be the topology generated by D_{μ}.

On $LH_{a,\mu,\alpha,m}$ we can define another set of norms associated with the original definition (2.7) of this space, namely

$$||\varphi||_{q,\delta}^{\mu} = \sup_{k,l} \sup_{\Omega_1} \frac{|x^k e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi\}|}{(m+\delta)^k \, k^{k\alpha}}$$

for $\varphi \in LH_{a,\mu,\alpha,m}$.

Let $T_{\mu,\delta}$ be the topology generated by the family of norms $D_{\mu,\delta} = \{ ||\varphi||_{q,\delta}^{\mu}$, $q = 0,1,\ldots$ and $\delta = 1, \frac{1}{2}, \ldots\}$. Then we have

THEOREM 3.23: The topologies T_{μ} and $T_{\mu,\delta}$ are equivalent and are finer than $T_{a,\mu,\alpha,m}$.

PROOF: For showing $T_{\mu} \approx T_{\mu,\delta}$, we need prove that D_{μ} and $D_{\mu,\delta}$ are equivalent.

Consider a member $||\cdot||_{q,\delta}^{\mu}$ in $D_{\mu,\delta}$, where

$$||\varphi||^{\mu}_{q,\delta} = \sup_{k,l} \sup_{\Omega_1} \frac{|x^k e^{at} D_t^l \Lambda^q \{x^{-(\frac{2\mu+1}{2})} \varphi\}|}{(m+\delta)^k k^{k\alpha}}$$

Let us note

$$\sup_k \left[\frac{x^k}{(m+\delta)^k k^{k\alpha}}\right] = \frac{1}{\inf f(k)} ,$$

where

$$f(k) = \frac{k^{k\alpha}}{T^k} , \quad T = \frac{x}{(m+\delta)} .$$

But

$$\frac{1}{\inf\limits_k f(k)} \leq \exp\left(\frac{\alpha}{e} T^{1/\alpha}\right)$$

by (3.20). Hence

$$||\varphi||^{\mu}_{q,\delta} \leq \sup_l \sup_{\Omega_1} \left[\exp\{\frac{\alpha}{e} (\frac{x}{m+\delta})^{1/\alpha} + at\}\right].$$

$$.\{ D_t^l \Delta^q \{x^{-(\frac{2\mu+1}{2})} \varphi\}|$$

$$\leq \sup_l \sup_{\Omega_1} \exp\{ \frac{\alpha}{em^{1/\alpha}} \frac{x^{1/\alpha}}{(1+\frac{\delta}{m})^{1/\alpha}} + at\} .$$

$$.|D_t^l \Delta^q \{x^{-(\frac{2\mu+1}{2})} \varphi\}|$$

Choose $p_o \in \mathbb{N}$ such that $\frac{1}{(1+\frac{\delta}{m})^{1/\alpha}} < 1 - \frac{1}{p_o}$.

Then for

$$b = \left(\frac{\alpha}{em^{1/\alpha}}\right),$$

$$||\varphi||^{\mu}_{q,\delta} \leq \sup_{l} \sup_{\Omega_1} \exp[b(1 - \frac{1}{p_o})x^{1/\alpha} + at] \cdot |D_t^l \Delta^q \{x^{-(\frac{2\mu+1}{2})} \varphi\}|$$

$$\leq \sup_{l} \sup_{\Omega_1} M_{p_o}(t,x) \,|\, D_t^l \, \Delta^q \{x^{-(\frac{2\mu+1}{2})} \varphi\}|$$

$$\leq \sup_{\substack{r\leq p \\ l\geq 0}} \sup_{\Omega_1} M_p(t,x) \,|\, D_t^l \, \Delta^r \{x^{-(\frac{2\mu+1}{2})} \varphi\}|$$

$$= ||\varphi||^{\mu}_{p}$$

where $p = \max(p_o, q)$

Conversely, for fixed $p \geq 2$ we have

$$(*) \quad ||\varphi||^{\mu}_{p} = \sup_{\substack{q\leq p \\ l\geq 0}} \sup_{\Omega_1} [\exp\{b(1 - \frac{1}{p})x^{1/\alpha} + at\} \cdot |D_t^l \, \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi\}|]$$

Again using (3.20), we get

$$(**) \quad \exp[\frac{\alpha}{e} \frac{x^{1/\alpha}}{(m+\delta)^{1/\alpha}}] \leq C_2 \cdot \sup_{k} [\frac{x^k}{(m+\delta)^k \, k^{k\alpha}}]$$

Since $0 < m[\frac{1}{(1-\frac{1}{p})^{\alpha}} - 1]$, we can choose $\delta > 0$ so that

$$1 - \frac{1}{p} < \frac{1}{(1+\frac{\delta}{m})^{1/\alpha}}.$$

Hence

$$+) \quad \exp[b(1 - \frac{1}{p})x^{1/\alpha}] < \exp[b \frac{m^{1/\alpha} x^{1/\alpha}}{(m+\delta)^{1/\alpha}}]$$

Consequently from (*) and (+),

$$||\varphi||_{p}^{\mu} \leq \sup_{\substack{q\leq p \\ l\geq 0}} \sup_{\Omega_1} \left[\exp\left\{ \frac{b\, m^{1/\alpha}\, x^{1/\alpha}}{(m+\delta)^{1/\alpha}}\right\} \cdot \left\{ e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi\}\right|\right]$$

$$\leq \sup_{\substack{q\leq p \\ l\geq 0}} \sup_{\Omega_1} C_2\left[\left\{ \sup_k \frac{x^k}{(m+\delta)^k\, k^{k\alpha}} \right\} \cdot \left| e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi\}\right|\right], \text{ by } (**)$$

$$\leq C_2 \sup_{q\leq p} ||\varphi||_{q,\delta}^{\mu} \;\#$$

A particular case of $LH_{a,\mu,\alpha,m}$ for $\alpha = 0$ is the space (2.13), as shown in

THEOREM 3.24 : $LH_{a,\mu,0,m} = LB_{a,\mu,m}$.

PROOF : For proving $LH_{a,\mu,0,m} \subset LB_{a,\mu,m}$, let $\varphi \in LH_{a,\mu,0,m}$. Then

$$(*) \qquad \left| x^k e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi\}\right| \leq C'_{ql}\, m^k$$

where $C'_{ql} = C_{ql} \cdot \frac{A^k}{m^k}$, C_{ql} as defined in (2.1) for $\alpha = 0$. Since

$$\inf_k \left(\frac{m}{x}\right)^k = \begin{cases} 1, & x \leq m \\ 0, & x > m \end{cases}$$

we infer from (*) by dividing it by x^k and then taking the infimum over k on R.H.S., that

$$\varphi(t,x) = 0 \text{ for } x > m.$$

As $\gamma^{\mu}_{a,q,l}(\varphi) < \infty$; $\varphi \in LB_{a,\mu,m}$.

For converse, consider $\varphi \in LB_{a,\mu,m}$. Then

$$| x^k e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi\} | = 0 \text{ for } x > m,$$

For $x \leq m$, let

$$C_{ql} = \sup_{(t,x) \in \mathbb{R}_+ \times (0,m)} | e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi\} | ,$$

where the right hand side exists from the definition of the space $LB_{a,\mu,m}$.

Then

$$\sup_{\Omega_1} | x^k e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi\} | \leq C'_{ql\delta} (m+\delta)^k$$

where $\quad C'_{ql\delta} = C_{ql} \dfrac{m^k}{(m+\delta)^k}$.

Thus $\varphi \in LH_{a,\mu,\alpha,m}$ and the result follows.

4. THE SPACES DEFINED ON Ω_2

This section initiates a discussion on spaces defined on Ω_2. The proofs of the results mentioned herein are analogous to the corresponding ones true for $LH_{a,\mu,\alpha}$ as discussed in section 3, and so omitted. It is also shown in this section that $LH_{a,\mu,\alpha}$ is homeomorphic to the space $\check{L}H_{a,\mu,\alpha}$ (cf. Theorem 4.5). To begin with, we have

THEOREM 4.1 : $(\overset{\vee}{LH}_{a,\mu,\alpha}, \overset{\vee}{T}_{a,\mu,\alpha})$ is a Fréchet space.

THEOREM 4.2 : For an even integer p in $\mathbb{N}$, $\overset{\vee}{LH}_{a,\mu+p,\alpha} \subset \overset{\vee}{LH}_{a,\mu,\alpha}$, and the topology on $\overset{\vee}{LH}_{a,\mu+p,\alpha}$ is stronger than the one induced on it by that of $\overset{\vee}{LH}_{a,\mu,\alpha}$, i.e. $\overset{\vee}{T}_{a,\mu,\alpha} \mid \overset{\vee}{LH}_{a,\mu+p,\alpha} \subset \overset{\vee}{T}_{a,\mu+p,\alpha}$.

THEOREM 4.3 : $D(\Omega_2)$ is a subspace of $\overset{\vee}{LH}_{a,\mu,\alpha}$, such that the injection map from $D(\Omega_2)$ to $\overset{\vee}{LH}_{a,\mu,\alpha}$ is continuous, i.e. $\overset{\vee}{T}_{a,\mu,\alpha} \mid D(\Omega_2) \subset T_{\Omega_2}$.

THEOREM 4.4 : The space $\overset{\vee}{LH}{}^{\beta}_{a,\mu,\alpha}$ is contained in the intersection of the spaces $\overset{\vee}{LH}_{a,\mu,\alpha}$ and $\overset{\vee}{LH}{}^{\beta}_{a,\mu}$. Further the topology $\overset{\vee}{T}{}^{\beta}_{a,\mu,\alpha}$ of $\overset{\vee}{LH}{}^{\beta}_{a,\mu,\alpha}$ coincides with the topologies $\overset{\vee}{T}_{a,\mu,\alpha} \mid \overset{\vee}{LH}{}^{\beta}_{a,\mu,\alpha}$ and $\overset{\vee}{T}{}^{\beta}_{a,\mu} \mid \overset{\vee}{LH}{}^{\beta}_{a,\mu,\alpha}$.

For the next result, observe that $\varphi(t,x) \in \overset{\vee}{LH}_{a,\mu,\alpha}$ whenever $\varphi(-t,x) \in LH_{a,\mu,\alpha}$. Indeed this defines an isomorphism

$$R : \overset{\vee}{LH}_{a,\mu,\alpha} \to LH_{a,\mu,\alpha}, \quad R(\varphi) = \overset{\vee}{\varphi},$$

where $\overset{\vee}{\varphi}(t,x) = \varphi(-t,x)$. Concerning this mapping R, we have

THEOREM 4.5 : The mapping R is a topological isomorphism from the space $(\overset{\vee}{LH}_{a,\mu,\alpha} ; \overset{\vee}{T}_{a,\mu,\alpha})$ onto the space $(LH_{a,\mu,\alpha} ; T_{a,\mu,\alpha})$.

PROOF : Let $\varphi \in LH_{a,\mu,\alpha}$ and $\gamma^{\mu}_{a,k,q,l} \in D_{a,\mu,\alpha}$.

Then,

$$\gamma^{\mu}_{a,k,q,l}(\overset{\vee}{\varphi}) = \sup_{t \in \Omega_1} | x^k e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi(-t,x) \} |$$

$$= \sup_{u \in \Omega_2} | x^k e^{-au} D_u^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi(u,x) \} |$$

$$\Rightarrow \gamma^{\mu}_{a,k,q,l}(\overset{\vee}{\varphi}) = \rho^{\mu}_{a,k,q,l}(\varphi).$$

Thus R is continuous. As isomorphism character of R and R^{-1} is trivial, the result follows.#

For the space $\overset{\vee}{LH}{}^{\beta}_{a,\mu}$ and $\overset{\vee}{LH}{}^{\beta}_{a,\mu,\alpha}$, we have similar result contained in

<u>THEOREM</u> 4.6 : The mapping R is a topological isomorphism from the space $\overset{\vee}{LH}{}^{\beta}_{a,\mu}$ (respectively $\overset{\vee}{LH}{}^{\beta}_{a,\mu,\alpha}$) onto the space $LH^{\beta}_{a,\mu}$ (respectively $LH^{\beta}_{a,\mu,\alpha}$).

5. <u>INDUCTIVE LIMITS</u>

In this section we confine our attention to the strict inductive limits of spaces studied in the previous sections. To begin with we have the following simple and useful

<u>PROPOSITION</u> 5.1 : If $m_1 < m_2$, then $LH_{a,\mu,\alpha,m_1} \subset LH_{a,\mu,\alpha,m_2}$ and $T_{a,\mu,\alpha,m_1} \approx T_{a,\mu,\alpha,m_2} | LH_{a,\mu,\alpha,m_1}$.

<u>PROOF</u> : For $\varphi \in LH_{a,\mu,\alpha,m_1}$

$$\gamma^{\mu}_{a,k,q,l}(\varphi) \leq C_{ql\delta}(m_1+\delta)^k k^{k\alpha}$$

$$\leq C_{ql\delta}(m_2+\delta)^k k^{k\alpha}.$$

Thus $LH_{a,\mu,\alpha,m_1} \subset LH_{a,\mu,\alpha,m_2}$. The second part is clear from the definition of topologies of these spaces.

We now define the inductive limit of the spaces $\{LH_{a,\mu,\alpha,m} : m \geq 1\}$ as follows :

<u>THEOREM</u> 5.2 : $LH_{a,\mu,\alpha} = \bigcup_{m=1}^{\infty} LH_{a,\mu,\alpha,m}$ and if the space $LH_{a,\mu,\alpha}$ is equipped with the strict inductive limit topology $S_{a,\mu,\alpha,m}$ defined by the injections from $LH_{a,\mu,\alpha,m}$ to $LH_{a,\mu,\alpha}$, then a sequence $\{\varphi_n\}$ in $LH_{a,\mu,\alpha}$ converges to zero if and only if $\{\varphi_n\}$ is contained in some $LH_{a,\mu,\alpha,m}$ and converges theirin to zero. Moreover, $LH_{a,\mu,\alpha}$ is complete.

<u>PROOF</u> : The above theorem is an immediate consequence of Theorem 1.2.5, once we show that $LH_{a,\mu,\alpha} = \bigcup_{m=1}^{\infty} LH_{a,\mu,\alpha,m}$.

Clearly $\bigcup_{m=1}^{\infty} LH_{a,\mu,\alpha,m} \subset LH_{a,\mu,\alpha}$.

For proving the other inclusion, let $\varphi \in LH_{a,\mu,\alpha}$.

Then

$$(*) \qquad \sup_{\Omega_1} |x^k e^{at} D_t^1 \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi\}| \leq C_{q1} A^k k^{k\alpha}$$

where A is some positive constant. Choose an integer $m \equiv m_A$ and a $\delta > 0$ such that

$$C_{q1} \; A^k \leq C_{q1\delta} (m_A+\delta)^k .$$

Then from (*), we immediately get $\varphi \in LH_{a,\mu,\alpha,m}$, implying that $LH_{a,\mu,\alpha} \subset \bigcup LH_{a,\mu,\alpha,m}$.#

REMARK : Analogous to Theorem 5.2 we can show that the spaces (2.2), (2.3), (2.4), (2.5) and (2.6) are the inductive limits of the corresponding spaces (2.8), (2.9), (2.10), (2.11) and (2.12), i.e.

$$LH^{\beta}_{a,\mu} = \bigcup_{n=1}^{\infty} LH^{\beta}_{a,\mu} \; ; \; LH^{\beta}_{a,\mu,\alpha} = \bigcup_{m,n=1}^{\infty} LH^{\beta,n}_{a,\mu,\alpha,m}$$
$$\overset{v}{LH}_{a,\mu,\alpha} = \bigcup_{m=1}^{\infty} \overset{v}{LH}_{a,\mu,\alpha,m} \; ; \; \overset{v}{LH}{}^{\beta}_{a,\mu} = \bigcup_{n=1}^{\infty} \overset{v}{LH}{}^{\beta,n}_{a,\mu}$$
$$\overset{v}{LH}{}^{\beta}_{a,\mu,\alpha} = \bigcup_{m,n=1}^{\infty} \overset{v}{LH}{}^{\beta,n}_{a,\mu,\alpha,m}.$$

Next we prove

PROPOSITION 5.3 : For real numbers a,b with $a < b$, $LH_{b,\mu,\alpha} \subset LH_{a,\mu,\alpha}$ and the induced topology on $LH_{b,\mu,\alpha}$ is weaker than its original topology, i.e.

$$T_{a,\mu,\alpha} \mid LH_{b,\mu,\alpha} \subset T_{b,\mu,\alpha}.$$

PROOF: Straight forward.

The above result suggests us to introduce

DEFINITION 5.4: Let $\{a_n\}$ be an strictly decreasing sequence converging to r, $-\infty < r < \infty$. Then we denote by $LH(r,\mu,\alpha)$, the space obtained from the increasing sequence of $\{LH_{a_n,\mu,\alpha} ; T_{a_n,\mu,\alpha}\}$ i.e.

$$LH(r,\mu,\alpha) = \bigcup_{n=1}^{\infty} LH_{a_n,\mu,\alpha}$$

Concerging this space we prove

THEOREM 5.5 : The space $LH(r,\mu,\alpha)$ is independent of the choice of sequence $\{a_n\}$. If $LH(r,\mu,\alpha)$ is equipped with the strict inductive limit topology defined by $LH_{a_n,\mu,\alpha}$ then a sequence $\{\varphi_n\}$ in $LH(r,\mu,\alpha)$ converges to zero if and only if $\{\varphi_n\}$ belongs to some $LH_{a_n,\mu,\alpha}$ and converges to zero in this space. Moreover, $LH(r,\mu,\alpha)$ is complete.

PROOF : In view of Theorem 1.2.5, we need prove the first part of this result.

Therefore consider two sequence $\{a_n\}$ and $\{b_n\}$ converging to r, to prove

$$\bigcup_{n=1}^{\infty} LH_{a_n,\mu,\alpha} = \bigcup_{n=1}^{\infty} LH_{b_n,\mu,\alpha},$$

let $\varphi \in \bigcup_{n=1}^{\infty} LH_{a_n,\mu,\alpha}$. Then there exists $m \in \mathbb{N}$ such that $\varphi \in LH_{a_m,\mu,\alpha}$. For this m, we can find b_1 such that $b_1 < a_m$. Hence $LH_{a_m,\mu,\alpha} \subset LH_{b_1,\mu,\alpha}$ by Proposition 5.3 and consequently

$$\bigcup_{n=1}^{\infty} LH_{a_n,\mu,\alpha} \subset \left(\bigcup_{n=1}^{\infty} LH_{b_n,\mu,\alpha}\right).$$

Similarly follows the other inclusion.#

Analogously we define the following spaces

$$LH^{\beta}(r,\mu) = \bigcup_{n=1}^{\infty} LH^{\beta}_{a_n,\mu}, \quad LH^{\beta}(r,\mu,\alpha) = \bigcup_{n=1}^{\infty} LH^{\beta}_{a_n,\mu,\alpha}$$

$$\overset{v}{LH}(r,\mu,\alpha) = \bigcup_{n=1}^{\infty} \overset{v}{LH}_{a_n,\mu,\alpha}, \quad \overset{v}{LH}^{\beta}(r,\mu) = \bigcup_{n=1}^{\infty} \overset{v}{LH}^{\beta}_{a_n,\mu}$$

$$\overset{v}{LH}^{\beta}(r,\mu,\alpha) = \bigcup_{n=1}^{\infty} \overset{v}{LH}^{\beta}_{a_n,\mu,\alpha}.$$

The last result of this section is

THEOREM 5.6: The space $D(\Omega_1)$ is dense in $LH(r,\mu,\alpha)$.

PROOF : Let φ be an arbitrary element of $LH(r,\mu,\alpha)$. Then $\varphi \in LH_{b,\mu,\alpha}$ for some $b > r$. Choose a such that $r < a < b$.

Let $A = (0,1]x(0,1]$ and V an open set containing A. By Proposition 1.4.3, there exists $\lambda(t,x) \in D(\Omega_1)$, $0 \leq \lambda(t,x) \leq 1$ such that

$$\lambda(t,x) = \begin{cases} 1 & \text{on } A \\ 0 & \text{out side } V \end{cases}$$

Define a sequence $\{\theta_n\} \in D(\Omega_1)$ by

$$\theta_n(t,x) = \lambda\left(\frac{t}{n}, \frac{x}{n}\right), \quad n \geq 1.$$

Clearly $\theta_n \varphi \in D(\Omega_1)$. Now consider

$$\gamma^{\mu}_{a,k,q,l}(\theta_n\varphi - \varphi) = \sup_{\Omega_1} | x^k e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} (\theta_n\varphi - \varphi)(t,x) \} |$$

$$= \sup_{\Omega_1} | x^k e^{at} D_t^l [\sum_{j=0}^{q} \binom{q}{j} \Delta^j \{ x^{-(\frac{2\mu+1}{2})}$$

$$. \varphi(t,x) \} . \Delta^{q-j} \{ (\theta_n - 1)(t,x) \}], \text{ by Lemma 3.17}$$

$$= \sup_{\Omega_1} | x^k e^{at} \sum_{i=0}^{l} \sum_{j=0}^{q} \binom{l}{i} \binom{q}{j} \{ D_t^i \Delta^j \{ x^{-(\frac{2\mu+1}{2})}$$

$$\varphi(t,x) \} . \{ D_t^{l-i} \Delta^{q-j} (\theta_n - 1)(t,x) \} |$$

$$(*) \qquad = \sup_{\Omega_1 \setminus A} |x^k e^{at} \sum_{i=0}^{l} \sum_{j=0}^{q} \binom{l}{i} \binom{q}{j} \{ D_t^i \Delta^j$$

$$. \{ x^{-(\frac{2\mu+1}{2})} \varphi(t,x)\} . \{ D_t^{l-i} \Delta^{q-j} (\theta_n - 1)(t,x)\}|$$

since $D_t^{l-i} \Delta^{q-j} (\theta_n - 1)$ is zero on A. Moreover

$$\sup_{\Omega_1 \setminus A} | x^k e^{at} D_t^i \Delta^j \{ x^{-(\frac{2\mu+1}{2})} \varphi(t,x) \}|$$

$$= \sup_{\Omega_1 \setminus A} | e^{(a-b)t} x^k e^{bt} D_t^i \Delta^j \{ x^{-(\frac{2\mu+1}{2})} \varphi \}|$$

$$\leq \gamma^{\mu}_{b,k,j,i}(\varphi) \{ e^{(a-b)t} \} .$$

Hence we have from (*)

$$\gamma^{\mu}_{a,k,q,l}(\theta_n \varphi - \varphi) \leq \sup_{\Omega_1 \setminus A} | \sum_{i=0}^{l} \sum_{j=0}^{q} \binom{l}{i} \binom{q}{j} \gamma^{\mu}_{b,k,j,i}(\varphi) .$$

$$. e^{(a-b)t} \{ D_t^{l-i} \Delta^{q-j} (\theta_n - 1)(t,x) \}|$$

As $\{D_t^{l-i} \Delta^{q-j} (\theta_n - 1)(t,x)\}$ is bounded on $\Omega_1 \setminus A$, the right hand side tends to zero as $n \to \infty$. Thus $\varphi \in \overline{D(\Omega_1)}$.#

CHAPTER - 4

LH - TRANSFORMATIONS

CONTENTS

1. INTRODUCTION :

This chapter is devoted to the study of various types of operators like dilations, multipliers etc. and the conventional Laplace - Hankel transformations, on spaces of the types LH. We prove several results on the continuity properties of these operators.

2. DILATION OPERATORS :

Before we start our discussion on dilation operators on spaces of type LH, let us observe

PROPOSITION 2.1 : For any given pair (ω, λ) of strictly positive numbers and $\varphi \in LH_{a,\mu,\alpha,m}$, the function $\Psi \in LH_{a\omega,\mu,\alpha,\frac{m}{\lambda}}$, where $\Psi(t,x) = \varphi(\omega t, \lambda x)$.

PROOF : Consider

$$\sup_{\Omega_1} | x^k e^{a\omega t} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \Psi(t,x) \} |$$

$$= \sup_{\Omega_1} | v^k e^{au} D_u^l (v^{-1} D_v)^q \{ v^{-(\frac{2\mu+1}{2})} \varphi(u,v) \} | \cdot \frac{\omega^l \lambda^{2q+(\frac{2\mu+1}{2})}}{\lambda^k}, \text{ for } u = \omega t,\ v = \lambda x$$

$$\text{(2.2)} \qquad \leq C_{ql\delta} (m+\delta)^k k^{k\alpha} \frac{\omega^l \lambda^{2q+(\frac{2\mu+1}{2})}}{\lambda^k}, \text{ since } \varphi \in LH_{a,\mu,\alpha,m}$$

$$\leq C'_{ql\delta} \left(\frac{m}{\lambda} + \delta'\right)^k k^{k\alpha},$$

Hence

$$\Psi \in LH_{a\omega,\mu,\alpha,\frac{m}{\lambda}} \ . \ \#$$

We are now in a position to introduce

DEFINITION 2.3 : For given pair (ω,λ) of strictly positive real numbers, an operator $R \equiv R_{\omega,\lambda} : (LH_{a,\mu,\alpha,m} ; T_{a,\mu,\alpha,m}) \to (LH_{a\omega,\mu,\alpha,\frac{m}{\lambda}} ; T_{a\omega,\mu,\alpha,\frac{m}{\lambda}})$ with $R\varphi = \Psi$, $\Psi(t,x) = \varphi(\omega t, \lambda x)$ is called a dilation operator.

NOTE : The operator R is clearly well defined and linear. More properties of R are given in

PROPOSITION 2.4 : The dilation operator R as defined above is a topological isomorphism from $LH_{a,\mu,\alpha,m}$ onto $LH_{a\omega,\mu,\alpha,\frac{m}{\lambda}}$.

PROOF : To prove 1-1 ness of R, let $R\varphi = 0$. Then $\varphi(u,v) = 0$ for $u = \omega t$, $v = \lambda x$, $(t,x) \in \Omega_1$. As $\omega,\lambda > 0$, clearly $\varphi(t,x) = 0$, $\forall\ (t,x) \in \Omega_1$.

Hence R is one-one

R is clearly surjective, for $\Psi \in LH_{a,\mu,\alpha,m}$ whenever $\varphi \in LH_{a\omega,\mu,\alpha,\frac{m}{\lambda}}$, where $\Psi(t,x) = \varphi(\frac{t}{\omega}, \frac{x}{\lambda})$.

The continuity of R is a consequence of the equation (2.2) of Proposition 2.1 ; indeed from (2.2) we infer

$$\gamma^{\mu}_{a\omega,k,q,1}(R\varphi) \leq M\ \gamma^{\mu}_{a,k,q,1}(\varphi).$$

The inverse mapping $R^{-1}: LH_{a\omega,\mu,\alpha,\frac{m}{\lambda}} \to LH_{a,\mu,\alpha,m}$, $R^{-1}(\Psi) = \varphi$,

$\varphi(t,x) = \Psi(\omega^{-1}t, \lambda^{-1}x)$; of R is a mapping of type R itself and so it is 1-1, onto and continuous.#

Similar to the above proposition, about the space $LH^{\beta,n}_{a,\mu}$ we have

PROPOSITION 2.5 : For any pair (ω,λ) of strictly positive real numbers and $\varphi \in LH^{\beta,n}_{a,\mu}$, the function $\Psi \in LH^{\beta,\lambda^2 n}_{a\omega,\mu}$, $\Psi(t,x) = \varphi(\omega t, \lambda x)$.

PROOF : For Ψ as above, consider

$$\sup_{I_1} | x^k e^{a\omega t} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \psi(t,x) \}|$$

$$= \sup_{I_1} | v^k e^{au} D_u^l (v^{-1}D_v)^q \{ v^{-(\frac{2\mu+1}{2})} \varphi(u,v)\}| .$$

$$\cdot \frac{\omega^l \lambda^{2q+(\frac{2\mu+1}{2})}}{\lambda^k}, \text{ for } u = \omega t,\ v = \lambda x$$

$$(2.6) \leq C'_{kl\eta} \{\lambda^2(n+\eta)\}^q q^{q\beta}$$

$$\leq C'_{kl\eta} (\lambda^2 n+\eta')^q q^{q\beta},\ \eta' > 0 ;$$

by definition of $LH^{\beta,n}_{a,\mu}$ and $\varphi \in LH^{\beta,n}_{a,\mu}$.

Consequently $\varphi(\omega t, \lambda x) \in LH^{\beta,\lambda^2 n}_{a\omega,\mu}$.#

PROPOSITION 2.7 : The dilation operator R :

$$(LH^{\beta,n}_{a,\mu} ; T^{\beta,n}_{a,\mu}) \to (LH^{\beta,\lambda^2 n}_{a\omega,\mu} ; T^{\beta,\lambda^2 n}_{a\omega,\mu})$$

with $R\varphi = \psi$, is a topological isomorphism

PROOF : Proof is analogous to that of Proposition 2.4. However for the continuity of R, use (2.6). #

3. MULTIPLIERS IN SPACES OF TYPE LH :

In this section we discuss operations of multiplication by exponential, power and smooth functions on spaces of the type LH. These operations are usually known as multipliers. We make a systematic study of these multipliers in three subsections of this section.

EXPONENTIAL MULTIPLIER :

We begin with

PROPOSITION 3.1 : Let $\sigma \in \mathbb{R}$ and $\varphi \in LH_{a-\sigma,\mu,\alpha}$. If $\psi(t,x) = \exp(-\sigma t)\, \varphi(t,x)$, then $\psi \in LH_{a,\mu,\alpha}$.

PROOF : For $\gamma^{\mu}_{a,k,q,l} \in D_{a,\mu,\alpha}$, consider

$$\gamma^{\mu}_{a,k,q,l}(\psi) = \sup_{\Omega_1} | x^k e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} (e^{-\sigma t} \varphi) \} |$$

$$= \sup_{\Omega_1} | x^k e^{at} \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \sum_{v=0}^{l} \binom{l}{v} \cdot$$

$$(-\sigma)^{l-v} e^{-\sigma t} D_t^v \varphi \} |$$

$$\leq \sum_{v=0}^{l} a_v \sup_{\Omega_1} | x^k e^{(a-\sigma)t} D_t^v \Delta^q .$$

$$. \{ x^{-(\frac{2\mu+1}{2})} \varphi \} , \quad a_v = \binom{l}{v} (-\sigma)^{l-v}$$

$$(3.2) \qquad \leq \sum_{v=0}^{1} a_v C_{qv} A^k k^{k\alpha}$$

since $\varphi \in LH_{a-\sigma,\mu,\alpha}$. Thus $\psi \in LH_{a,\mu,\alpha}$.

In view of the above proposition, it is possible to introduce.

DEFINITION 3.3 : An operator $S : LH_{a-\sigma,\mu,\alpha} \to LH_{a,\mu,\alpha}$ with $S\varphi = \psi$ is known as an exponential multiplier.

REMARK : The operator S is clearly well defined and linear.

Concerning these multipliers we have

PROPOSITION 3.4 : Each exponential multiplier $S:(LH_{a-\sigma,\mu,\alpha} ; T_{a-\sigma,\mu,\alpha}) \to (LH_{a,\mu,\alpha} ; T_{a,\mu,\alpha})$ is a topological isomorphism.

PROOF: S is injective, since exponential function never assumes the value zero.

Also, $\psi \in LH_{a-\sigma,\mu,\alpha}$, $\psi(t,x) = e^{\sigma t} \varphi(t,x)$ whenever $\varphi \in LH_{a,\mu,\alpha}$. Hence S is surjective too.

For the continuity of S, note the following from (3.2)

$$\gamma^{\mu}_{a,k,q,l}(e^{-\sigma t}\varphi) \leq \sum_{v=0}^{1} a_v \gamma^{\mu}_{a-\sigma,k,q,v}(\varphi).$$

The inverse mapping $S^{-1} : LH_{a,\mu,\alpha} \to LH_{a-\sigma,\mu,\alpha}$ defined by $S^{-1}(\varphi) = \psi$, $\psi(t,x) = e^{\sigma t} \varphi(t,x)$ can be proved to be continuous 1-1 and onto in a similar way. #

Concerning the adjoint S^* of S, Proposition 3.4 and Proposition 1.3.3 immediately yield.

PROPOSITION 3.5 : The adjoint S^* of S is a $\beta(LH^*_{a,\mu,\alpha}, LH_{a,\mu,\alpha}) - \beta(LH^*_{a-\sigma,\mu,\alpha} ; LH_{a-\sigma,\mu,\alpha})$ isomorphism from $LH^*_{a,\mu,\alpha}$ onto $LH^*_{a-\sigma,\mu,\alpha}$.

POWER MULTIPLIERS :

For introducing the notion of power multiplier operators, we need to prove

PROPOSITION 3.6 : For $\varphi \in LH_{a,\mu,\alpha}$ and $n \in \mathbb{Z}$ if $\Psi(t,x) = x^n \varphi(t,x)$, $(t,x) \in \Omega_1$, then $\Psi \in LH_{a,\mu+n,\alpha}$.

PROOF : Observe that

$$\sup_{\Omega_1} | x^k e^{at} D_t^1 \Delta^q \{ x^{-(\frac{2\mu+2n+1}{2})} x^n \varphi \} |$$

$$= \gamma^{\mu}_{a,k,q,1} (\varphi)$$

$$\leq C_{q1} A^k k^{k\alpha}$$

since $\varphi \in LH_{a,\mu,\alpha}$. Thus $\Psi \in LH_{a,\mu+n,\alpha}$. Moreover, it is clear from above that

$$(*) \qquad \gamma^{\mu+n}_{a,k,q,1}(\Psi) = \gamma^{\mu}_{a,k,q,1}(\varphi)$$

The above result helps us to introduce

DEFINITION 3.7 : For $n \in \mathbb{Z}$ an operator $P: LH_{a,\mu,\alpha} \to LH_{a,\mu+n,\alpha}$

defined by $P\varphi = \Psi$, where $\Psi(t,x) = x^n \varphi(t,x), \varphi \in LH_{a,\mu,\alpha}$, $(t,x) \in \Omega_1$; is called a power multiplier.

Concerning power multiplier operators, we have

PROPOSITION 3.8 : The power operator P defined as in Definition 3.7 is a topological isomorphism.

PROOF : Clearly, P is well defined linear and injective.

It is surjective because for Ψ in $LH_{a,\mu+n,\alpha}$, $\varphi(t,x) = x^{-n}\Psi(t,x)$ is a member of $LH_{a,\mu,\alpha}$ such that $P(\varphi) = \Psi$.

Continuity of P is immediate from (*).

The inverse mapping $P^{-1} : LH_{a,\mu+n,\alpha} \to LH_{a,\mu,\alpha}$ is given by $P^{-1}\Psi = \varphi$, $\varphi(t,x) = x^{-n}\Psi(t,x)$ and so it is clearly linear, bijective and continuous. #

The above proposition along with Theorem 1.3.3, yields

PROPOSITION 3.9 : Let $P^*: LH^*_{a,\mu+n,\alpha} \to LH^*_{a,\mu,\alpha}$ be the adjoint of the map P. Then P^* is $\beta(LH^*_{a,\mu+n,\alpha}, LH_{a,\mu+n,\alpha})-\beta(LH^*_{a,\mu,\alpha}, LH_{a,\mu,\alpha})$ isomorphism.

SMOOTH MULTIPLIER :

For the definition of such multiplier operators, let us recall the spaces Ⓗ and J from section 7 of Chapter 1, namely Definition 1.7.1 and 1.7.2. In this direction we first prove

PROPOSITION 3.10 : Let $a,b \in \mathbb{R}$ with $a < b$ and $\theta \in$ Ⓗ. Then

the product $\theta\varphi \in LH_{a,\mu,\alpha}$, for each φ in $LH_{b,\mu,\alpha}$.

PROOF : Since $a < b$ and $\theta \in (H)$, for some constant $C_{l\nu}$ we get

$$(*) \qquad \left| \{ e^{(a-b)t} (1+t^2)^{N_{l-\nu}} \} \{ \frac{D^{l-\nu}\theta}{(1+t^2)^{N_{l-\nu}}} \} \right| < C_{l\nu} .$$

Consider

$$\sup_{\Omega_1} | x^k e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \theta(t) \varphi(t,x) \} |$$

$$= \sup_{\Omega_1} | x^k e^{bt} \Delta^q x^{-(\frac{2\mu+1}{2})} e^{(a-b)t} . \sum_{\nu=0}^{l} \binom{l}{\nu} (D_t^{l-\nu}\theta)(D_t^{\nu}\varphi) |$$

$$= \sup_{\Omega_1} | \sum_{l=0}^{\nu} \binom{l}{\nu} [x^k e^{bt} D_t^{\nu} \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi \}] . [e^{(a-b)t} (1+t^2)^{N_{l-\nu}} \frac{D_t^{l-\nu}\theta(t)}{(1+t^2)^{N_{l-\nu}}}]$$

$$\leq \sum_{l=0}^{\nu} \binom{l}{\nu} C_{l\nu} C_{q\nu} A^k k^{k\alpha}$$

by (*) and using the definition of members of $LH_{b,\mu,\alpha}$.

Thus $\theta\varphi \in LH_{a,\mu,\alpha}$ and also

$$(**) \qquad \gamma^{\mu}_{a,k,q,l}(\theta\varphi) \leq \sum_{l=0}^{\nu} \binom{l}{\nu} C_{l\nu} \gamma^{\mu}_{b,k,q,l}(\varphi).$$

For member of J , we have

PROPOSITION 3.11 : Let $\varphi \in LH_{a,\mu,\alpha}$ and $\Psi \in J$. Then $\Psi\varphi \in LH_{a,\mu,\alpha}$

PROOF : Using the definition of J, for $\nu \in \mathbb{N}$, there exists $N_\nu \in \mathbb{N}$ and a constant M_ν such that

$$\frac{\Delta^{\nu}\,\psi(x)}{1+x^{N_\nu}} < M_\nu$$

Therefore

$$\sup_{\Omega_1} | x^k e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \psi(x)\, \varphi(t,x) \} |$$

$$= \sup_{\Omega_1} | x^k e^{at} D_t^l [\{ \sum_{\nu=0}^{q} \binom{q}{\nu} \frac{\Delta^\nu \psi}{1+x^{N_\nu}} (1+x^{N_\nu}) \Delta^{q-\nu} \{ x^{-(\frac{2\mu+1}{2})} \varphi \}]$$

$$\leq \sum_{\nu=0}^{q} \sup_{\Omega_1} | x^k e^{at} D_t^l (1+x^{N_\nu}) \Delta^{q-\nu} \{ x^{-(\frac{2\mu+1}{2})} \varphi \} | \binom{q}{\nu} M_\nu$$

$$\leq \sum_{\nu=0}^{q} M_\nu \binom{q}{\nu} [\gamma^{\mu}_{a,k,q-\nu,l}(\varphi) + \gamma^{\mu}_{a,k+N_\nu,q-\nu,l}(\varphi)]$$

Hence $\Psi\varphi \in LH_{a,\mu,\alpha}$ and

$$(*) \qquad \gamma^{\mu}_{a,k,q,l}(\Psi\varphi) \leq \sum_{\nu=0}^{q} M_\nu \binom{q}{\nu} [\gamma^{\mu}_{a,k,q-\nu,l}(\varphi) + \gamma^{\mu}_{a,k+N_\nu,q-\nu,l}(\varphi)].$$

These propositions lead us to

DEFINITION 3.12 : For a,b in $\mathbb{R}$ with $a < b$, and $\theta \in (H)$ (respectively $\Psi \in J$) the operator $S : LH_{b,\mu,\alpha} \to LH_{a,\mu,\alpha}$ with $S\varphi = \theta\varphi$ (respectively $S : LH_{a,\mu,\alpha} \to LH_{a,\mu,\alpha}$ with $S\varphi = \Psi\varphi$) is called a smooth multiplier operator.

These multiplier operators satisfy

PROPOSITION 3.13: The smooth multiplier $S: (LH_{b,\mu,\alpha}; T_{b,\mu,\alpha}) \to (LH_{a,\mu,\alpha}; T_{a,\mu,\alpha})$ [respectively $(LH_{a,\mu,\alpha}; T_{a,\mu,\alpha}) \to (LH_{a,\mu,\alpha}; T_{a,\mu,\alpha})$] as defined above in Definition 3.12 is a continuous linear transformation.

PROOF : S is clearly well defined and linear. Moreover, it is continuous by inequality (**) of Proposition 3.10 [respectively (*) of Proposition 3.11].

4. DIFFERENTIAL AND INTEGRAL OPERATORS :

This section includes the study of three types of differential and integral operators on spaces of the type LH, defined below.

For $\varphi \in E(\Omega_1)$ and $\mu \in \mathbb{R}$

$$(4.1) \qquad N_\mu \varphi(t,x) = x^{(\frac{2\mu+1}{2})} D_x \{x^{-(\frac{2\mu+1}{2})} \varphi(t,x)\}$$

$$(4.2) \qquad N_\mu^{-1} \varphi(t,x) = -x^{(\frac{2\mu+1}{2})} \int_x^\infty v^{-(\frac{2\mu+1}{2})} \varphi(t,v)\,dv$$

$$(4.3) \qquad M_\mu \varphi(t,x) = x^{-(\frac{2\mu+1}{2})} D_x \{x^{(\frac{2\mu+1}{2})} \varphi(t,x)\}$$

$$(4.4) \qquad L_\mu \varphi(t,x) = M_\mu N_\mu \varphi(t,x) = [D_x^2 - (\frac{4\mu^2-1}{4x^2})]\, \varphi(t,x)$$

We begin with the study of the operator N_μ and prove

PROPOSITION 4.5 : For $\varphi \in LH_{a,\mu,\alpha,m}$, $N_\mu \varphi \in LH_{a,\mu+1,\alpha,m}$ and the map $N_\mu : (LH_{a,\mu,\alpha,m} ; T_{a,\mu,\alpha,m}) \to (LH_{a,\mu+1,\alpha,m} ; T_{a,\mu+1,\alpha,m})$ is one-one linear and continuous.

PROOF : Let $\varphi \in LH_{a,\mu,\alpha,m}$. Then

$$\sup_{\Omega_1} | x^k e^{at} D_t^l \Delta^q \{x^{-(\frac{2(\mu+1)+1}{2})} N_\mu \varphi \}|$$

$$= \sup_{\Omega_1} | x^k e^{at} D_t^l \Delta^q [x^{-1} x^{-(\frac{2\mu+1}{2})} \{ x^{(\frac{2\mu+1}{2})} D_x (x^{-(\frac{2\mu+1}{2})} \varphi) \}] |$$

$$= \sup_{\Omega_1} | x^k e^{at} D_t^l \Delta^{q+1} \{x^{-(\frac{2\mu+1}{2})} \varphi \}|$$

$$\leq C'_{ql\delta} (m+\delta)^k k^{k\alpha} ,$$

(cf. Definition 3.2.7). Thus $N_\mu(\varphi) \in LH_{a,\mu+1,\alpha,m}$ and therefore operator N_μ is well defined. It is obviously linear. It is injective, for

$N_\mu \varphi = 0$

$$\Rightarrow \quad D_x[x^{-(\frac{2\mu+1}{2})} \varphi (t, x)] = 0$$

$$\Rightarrow \quad x^{-(\frac{2\mu+1}{2})} \varphi(t,x) = C(t),$$

where $C(t)$ is a constant with respect to x. If $C(t) = 0$, $\varphi = 0$; otherwise $\varphi = C(t)\ x^{(\frac{2\mu+1}{2})}$

$$\Rightarrow \quad \sup_{\Omega_1} |x^k e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi\}| = \sup_{\Omega_1} |x^k e^{at} C(t) | ,$$

for $q = l = 0$.

As the right hand side of the above equation is not bounded, we conclude $\varphi \notin LH_{a,\mu,\alpha,m}$ which is a contradiction. Hence $C(t) = 0$, that is $\varphi = 0$.

This proves that N_μ is injective. For the continuity at N_μ, observe that

$$\gamma^{\mu+1}_{a,k,q,l}(N_\mu \varphi) \leq M \, \gamma^{\mu}_{a,k,q,l}(\varphi)$$

M is some constant.#

Next we have

<u>PROPOSITION</u> 4.6 : The operator N_μ^{-1} as defined in (4.2) is a continuous linear one-one map from $(LH_{a,\mu+1,\alpha,m} ; T_{a,\mu+1,\alpha,m})$ to $(LH_{a,\mu,\alpha,m} ; T_{a,\mu,\alpha,m})$.

<u>PROOF</u> : We first show that $N_\mu^{-1} \varphi \in LH_{a,\mu,\alpha,m}$ whenever $\varphi \in LH_{a,\mu+1,\alpha,m}$.

Let us therefore take φ in $LH_{a,\mu+1,\alpha,m}$. Then

$$\sup_{\Omega_1} | x^k e^{at} D_t^l \Delta^q \{ x^{-(\frac{2\mu+1}{2})} N_\mu^{-1} \varphi \} |$$

$$= \sup_{\Omega_1} | x^k e^{at} D_t^l \Delta^{q-1} x^{-1} [D_x \{ - \int_x^\infty v^{-(\frac{2\mu+1}{2})} \varphi(t,v) dv \}] |$$

$$= \sup_{\Omega_1} | x^k e^{at} D_t^l \Delta^{q-1} [x^{-(\frac{2(\mu+1)+1}{2})} \varphi(t,x)] |$$

$$\leq C'_{ql\delta} (m+\delta)^k k^{k\alpha}, \text{ by } 3.2.7.$$

Hence $N_{\mu}^{-1}(\varphi) \in LH_{a,\mu,\alpha,m}$. Moreover, it is clear from above that

$$\gamma^{\mu}_{a,k,q,l}(N_{\mu}^{-1}\varphi) = \gamma^{\mu+1}_{a,k,q-1,l}(\varphi)$$

Thus N_{μ}^{-1} is continuous map from $(LH_{a,\mu+1,\alpha,m} ; T_{a,\mu+1,\alpha,m})$ to $(LH_{a,\mu,\alpha,m} ; T_{a,\mu,\alpha,m})$. Clearly N_{μ}^{-1} is linear and injective.

Combining the above two poropositions, we get

THEOREM 4.7 : The operator N_{μ} (respectively N_{μ}^{-1}) is a topological isomorphism from the space $LH_{a,\mu,\alpha,m}$ (respectively $LH_{a,\mu+1,\alpha,m}$) onto the space $LH_{a,\mu+1,\alpha,m}$ (respectively $LH_{a,\mu,\alpha,m}$.

PROOF : For the truth of the result, we have to check that $N_{\mu} o N_{\mu}^{-1} = I$ and $N_{\mu}^{-1} o\, N_{\mu} = I$ which trivially follows from the definitions of these operators.#

An immediate consequence of Theorem 4.7 and Theorem 1.3.3 is

THEOREM 4.8 : Let $N_{\mu}^{*} : LH^{*}_{a,\mu+1,\alpha,m} \rightarrow LH^{*}_{a,\mu,\alpha,m}$ be the adjoint of the map $N_{\mu} : LH_{a,\mu,\alpha,m} \rightarrow LH_{a,\mu+1,\alpha,m}$. Then N_{μ}^{*} is a $\beta(LH^{*}_{a,\mu+1,\alpha,m}, LH_{a,\mu+1,\alpha,m}) - \beta(LH^{*}_{a,\mu,\alpha,m}, LH_{a,\mu,\alpha,m})$ isomorphism.

For the operator M_{μ}, we have

PROPOSITION 4.9 : For $\varphi \in LH_{a,\mu+1,\alpha,m}$, $M_{\mu}\varphi \in LH_{a,\mu,\alpha,m}$ and the map $M_{\mu} : (LH_{a,\mu+1,\alpha,m} ; T_{a,\mu+1,\alpha,m}) \rightarrow (LH_{a,\mu,\alpha,m}, T_{a,\mu,\alpha,m})$

is a continuous linear transformation.

PROOF : Proceeding on the lines of proof of Proposition 4.5, we get the result.#

From what has been proved in this section, it follows :

THEOREM 4.10 : The operator $L_\mu : (LH_{a,\mu,\alpha,m} ; T_{a,\mu,\alpha,m}) \to (LH_{a,\mu,\alpha,m} ; T_{a,\mu,\alpha,m})$ defines a continuous linear transformation.

5. THE CONVENTIONAL LAPLACE-HANKEL TRANSFORM:

In this section we initiate the study of Laplace-Hankel transforms on spaces of the type LH, as defined in

DEFINITION 5.1 : The Laplace-Hankel transform, denoted by $\Phi(s,y)$ or $lh_\mu \varphi(t,x)$ of a function φ in $LH_{a,\mu,\alpha,m}$, is an operator from $\mathbb{C} \times (o\infty)$ to $\mathbb{C}$ given by

$$\Phi(s,y) \equiv lh_\mu \{\varphi(t,x)\}$$

$$(5.2) \qquad = \int_o^\infty \int_o^\infty e^{-st} \sqrt{xy}\, J_\mu(xy)\, \varphi(t,x)\, dx\, dt$$

where $J_\mu(x)$ is the Bessel function of first kind of order μ with $2\mu+1 \geq 0$ and $Re(s) > (-a)$.

DEFINITION 5.3 : The Laplace-Hankel transform, denoted by $\Phi(s,y)$ or $lh_\mu^- \varphi(t,x)$ of a function φ in $\overset{\vee}{LH}_{a,\mu,\alpha,m}$, is an operator from $\mathbb{C} \times (o,\infty)$ to $\mathbb{C}$ given by

$$\Phi(s,y) = lh_\mu^- \{\varphi(t,x)\}$$

$$= \int_{-\infty}^o \int_o^\infty e^{-st} \sqrt{xy}\, J_\mu(xy)\, \varphi(t,x)\, dx\, dt$$

where $J_\mu(x)$ is as given in Definition 5.1 and $Re(s) < a$. The inversion formula for (5.2) is usually given by

$$(5.4)\quad \varphi(t,x) = \frac{1}{2\pi i} \int_0^\infty \sqrt{xy}\; J_\mu(xy) \int_{\sigma - i\infty}^{\sigma + i\infty} e^{st}\, \Phi(s,y)\,ds\,dy$$

The proofs of the main results of this section (Theorem 5.9 and 5.10), which exhibit continuity properties of these transforms require.

LEMMA 5.5: For $\varphi \in LH_{a,\mu,\alpha,m}$,

$$N_{\mu+q+k-1} \cdots N_{\mu+q}\, x^q \varphi(t,x)$$

$$= x^q N_{\mu+k-1} \cdots N_\mu \varphi(t,x)$$

where N_μ is defined as in (4.1).

PROOF: Result is immediate by induction.#

LEMMA 5.6 : If $\varphi \in LH_{a,\mu,\alpha,m}$ and

$$\Phi(s,y) = lh_\mu \varphi(t,x)\ ,\ \text{then}$$

$$\text{(i)}\qquad N_{\mu+k-1} \cdots N_\mu\, \varphi(t,x) = x^{\left(\frac{2\mu+2k+1}{2}\right)} \Delta^k \left[x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t,x)\right]$$

$$\text{(ii)}\qquad N_{\mu+q-1} \cdots N_\mu\, \Phi(s,y) = y^{\left(\frac{2\mu+2q+1}{2}\right)} (y^{-1}D_y)^q \left[y^{-\left(\frac{2\mu+1}{2}\right)} \Phi(s,y)\right]$$

PROOF : (i) Follows immediately by using induction.

(ii) Analogous to (i), indeed, replace t, x and φ in (i) by s, y and Φ respectively.#

LEMMA 5.7 : Let $\varphi \in LH_{a,\mu,\alpha,m}$ and $2\mu+1 \geq 0$.

Then

(i) $\quad lh_{\mu+1}(-x\,\varphi) = N_\mu\, lh_\mu\, \varphi(t,x) \equiv N_\mu\, \Phi(s,y)$

(ii) $\quad lh_{\mu+1}(N_\mu\, \varphi) = -y\, lh_\mu \varphi$

PROOF : (i) Consider

$N_\mu\, \Phi(s,y)$

$$= y^{(\frac{2\mu+1}{2})} D_y\, y^{-(\frac{2\mu+1}{2})} \int_0^\infty \int_0^\infty e^{-st} \sqrt{xy}\, J_\mu(xy)\, \varphi(t,x)\, dx\, dt$$

$$= y^{(\frac{2\mu+1}{2})} \int_0^\infty \int_0^\infty e^{-st} \sqrt{x}\, \varphi(t,x)\, D_y\, [y^{-\mu} J_\mu(xy)]\, dxdt$$

$$= y^{(\frac{2\mu+1}{2})} \int_0^\infty \int_0^\infty e^{-st} \sqrt{x}\, \varphi(t,x)\, [-xy^{-\mu} \cdot J_{\mu+1}(xy)]\, dxdt$$

since $D_y\, y^{-\mu} J_\mu(xy) = -xy^{-\mu} J_{\mu+1}(xy)$ (cf. [49] ,p.154)

Hence

$N_\mu\, \Phi(s,y) = lh_{\mu+1}(-x\,\varphi)$ and (i) follows.

(ii) Consider

$$lh_{\mu+1}(N_\mu \varphi) = \int_0^\infty \int_0^\infty e^{-st} \sqrt{xy}\, J_{\mu+1}(xy)\, \{N_\mu\, \varphi(t,x)\}\, dxdt$$

$$= \int_0^\infty \int_0^\infty e^{-st} \sqrt{xy}\, J_{\mu+1}(xy)\, x^{\frac{2\mu+1}{2}} \{D_x (x^{-(\frac{2\mu+1}{2})} \varphi(t,x)\}\, dxdt$$

$$= \int_0^\infty \int_0^\infty \sqrt{y}\, e^{-st} \{x^{\mu+1} J_{\mu+1}(xy)\} \{D_x (x^{-(\frac{2\mu+1}{2})} \varphi(t,x)\}\, dxdt$$

$$= - y \, 1h_{\mu} \, \varphi(t,x)$$

by integrating by parts and using the result $D_x [x^{\mu+1} J_{\mu+1}(xy)] = y \, x^{\mu+1} J_{\mu}(xy)$ (cf. [51] , p.154). #

LEMMA 5.8 : Consider $\Phi(s,y)$ for $\varphi \in LH_{a,\mu,\alpha,m}$. Then

$(-y)^k N_{\mu+q-1} \cdots N_{\mu} \Phi(s,y)$

$$= 1h_{\mu+k+q} [(-x)^q N_{\mu+k-1} \cdots N_{\mu} \varphi(t,x)]$$

PROOF : From Lemma 5.7(i), $N_{\mu} \Phi = 1h_{\mu+1}(-x\varphi)$.

Therefore if $\Psi = -x\varphi$, then

$$N_{\mu+1}(N_{\mu} \Phi) = N_{\mu+1}(1h_{\mu+1} \Psi) = 1h_{\mu+2}(-x \Psi)$$

again using Lemma 5.7(i).

Repeating this process q times and using Lemma 5.7(i) we get

$$N_{\mu+q-1} \cdots N_{\mu} \Phi = 1h_{\mu+q}\{(-x)^q \varphi\}.$$

Multiplying both sides by $-y$ and writing $\Psi = (-x)^q \varphi$, we have

$$\begin{aligned}(-y) [N_{\mu+q-1} \cdots N_{\mu} \Phi] &= -y \, 1h_{\mu+q} [\Psi] \\ &= 1h_{\mu+q+1} N_{\mu+q}(\Psi)\end{aligned}$$

by Lemma 5.7(ii). Hence

$$(-y) [N_{\mu+q-1} \cdots N_{\mu} \Phi] = 1h_{\mu+q+1} N_{\mu+q} [(-x)^q \varphi]$$

Again repeating this k times and using Lemma 5.7(ii) we get

$$\begin{aligned}&(-y)^k [N_{\mu+q-1} \cdots N_{\mu} \Phi] \\ &\quad = 1h_{\mu+q+k} [N_{\mu+k+q-1} \cdots N_{\mu+q}\{(-x)^q \varphi\}]\end{aligned}$$

$$= lh_{\mu+q+k}[\,(-x)^q\, N_{\mu+k-1} \cdots N_{\mu}\, \varphi\,]$$

by Lemma 5.5.#

We are now prepared to prove the main results contained in

THEOREM 5.9 : Let a, s be two real numbers with $a < 0$, $s > 0$ and $s+a > 0$. Then the conventional Laplace-Hankel transform lh_μ from $(LH_{a,\mu,\alpha,m}\,;\, T_{a,\mu,\alpha,m})$ into the space $(LH_{a,\mu}^{2\alpha,(2e)^{2\alpha} m^2}\,;\, T_{a,\mu}^{2\alpha,(2e)^{2\alpha} m^2})$ is a continuous linear mapping.

PROOF : Let us first prove that $\Phi \equiv lh_\mu \varphi \in LH_{a,\mu}^{2\alpha,(2e)^{2\alpha} m^2}$ whenever $\varphi \in LH_{a,\mu,\alpha,m}$. Therefore, for $k, q, l = 0, 1, \ldots,$ consider

$$E = |e^{as} D_s^l\, y^k\, (y^{-1} D_y)^q \{ y^{-(\frac{2\mu+1}{2})} \Phi(s,y)\}|$$

$$= |e^{as} D_s^l\, y^{-(\mu+q+\frac{1}{2})} (-y)^k \{N_{\mu+q-1} \cdots N_\mu \Phi\}|,$$

(cf. Lemma 5.6(ii))

$$= |e^{as} D_s^l\, y^{-(\mu+q+\frac{1}{2})}\, lh_{\mu+k+q}\{\,(-x)^q\, N_{\mu+k-1} \cdots N_\mu \varphi\}|,$$

(cf. Lemma 5.8)

$$= |e^{as} D_s^l\, y^{-(\mu+q+\frac{1}{2})} \int_0^\infty \int_0^\infty e^{-st} (-x)^q\, x^{\mu+k+\frac{1}{2}}$$

$$\Delta^q \{ x^{-(\frac{2\mu+1}{2})} \varphi\} \sqrt{xy}\; J_{\mu+k+q}(xy)\, dx\, dt|$$

by Lemma 5.6(i). Since $\left|\frac{J_{\mu+k+q}(xy)}{(xy)^{\mu+q}}\right| \leq M_1$, by some constant $M_1 > 0$ ([51],p.134), we get

$$E \leq M_1 \int_0^\infty \int_0^\infty |e^{as} D_s^l e^{-st} x^{2\mu+2q+k+1} \Delta^k, \{x^{-(\frac{2\mu+1}{2})} \varphi\} dxdt,$$

$$= M_1 \int_0^\infty \int_0^\infty |e^{as-st} (t)^l \{D_t^{-l} e^{-at} e^{at} D_t^l\}.$$

$$. x^{v+k+2q}(\frac{1+x^2}{1+x^2}) \Delta^k \{x^{-(\frac{2\mu+1}{2})} \varphi\}| dxdt$$

$$(*) = M_1 \int_0^\infty \int_0^\infty |\{e^{as-st} t^l \frac{e^{-at}}{a^l} \cdot \frac{1}{1+x^2}\}.\{(x^{v+k+2q}+x^{v+k+2q+2}).$$

$$. e^{at} D_t^l \Delta^k (x^{-(\frac{2\mu+1}{2})} \varphi)\}| dxdt$$

$$\leq M_1 \int_0^\infty \int_0^\infty | \frac{e^{as}}{a^l} t^l e^{-(s+a)t} \frac{1}{1+x^2} | dxdt.$$

$$. [C_{kl\delta} (m+\delta)^{v+k+2q} (v+k+2q)^{(v+k+2q)\alpha} +$$

$$C_{kl\delta}(m+\delta)^{v+k+2q+2} (v+k+2q+2)^{(v+k+2q+2)\alpha}]$$

Let us estimate

$$(m+\delta)^{v+k+2q} (v+k+2q)^{(v+k+2q)\alpha}$$

$$= C_1 (m+\delta)^{2q} (v+k+2q)^{(v+k)\alpha} \cdot (v+k+2q)^{2q\alpha},\ C_1 = (m+\delta)^{v+k}$$

$$= C_2(m^2+2m\,\delta+\delta^2)^q \{1+\frac{2q}{v+k}\}^{(v+k)\alpha} (2q)^{2q\alpha} \{1+\frac{v+k}{2q}\}^{2q\alpha},$$

$$C_2 = C_1(v+k)^{(v+k)\alpha}$$

$$\leq C_2(m^2+\delta')^q\, e^{2q\alpha}\, (2q)^{2q\alpha}\, e^{(v+k)\alpha},\quad \delta' = 2m\delta+\delta^2$$

$$\leq C_3\{(2e)^{2\alpha}\, m^2 + \delta''\}^q\, q^{q(2\alpha)}$$

where $C_3 = C_2\, e^{(v+k)\alpha}$, a constant relative to q and $\delta'' = \delta'(2e)^{2\alpha} > 0$.

Also

$$(m+\delta)^{v+k+2q+2}(v+k+2q+2)^{(v+k+2q+2)\alpha}$$

$$\leq C_5\{(2e)^{2\alpha}\, m^2+\delta''\}^q\, q^{q(2\alpha)}$$

where C_5 is a constant relative to q and $\delta'' > 0$.

Observe that

$$M_1 \int_0^\infty \int_0^\infty \left| \frac{e^{as}}{a^1}\, t^1\, e^{-(s+a)t}\, \frac{1}{1+x^2}\right| dxdt$$

$$= M_1\, \frac{\pi}{2} \int_0^\infty \left| \frac{e^{as}}{a^1}\, t^1\, e^{-(s+a)t}\right| dt$$

$\leq C$ where C is a constant.

Hence

$$E \leq C(C_3+C_5)\, C_{kl\delta}\{(2e)^{2\alpha}\, m^2+\delta''\}^q\, q^{q2\alpha}$$

$$\leq C'_{kl\delta}\ \{(2e)^{2\alpha}\, m^2+\delta''\}^q\, q^{q(2\alpha)}$$

Thus $\Phi \in LH^{2\alpha,(2e)^{2\alpha} m^2}_{a,\mu}$ and therefore the mapping $\varphi \to \Phi$ is well defined. It is obviously linear. Its continuity is a consequence of

$$\gamma^{\mu}_{a,k,q,1}(\Phi) \le M_3 \{ \gamma^{\mu}_{a,v+k+2q,k,1}(\varphi) + \gamma^{\mu}_{a,v+k+2q+2,k,1}(\varphi)$$

where M_3 is some constant ; which we derive from (*). #

Final result of this section is

THEOREM 5.10 : For a, s as in Theorem 5.9, the conventional Laplace-Hankel transform lh_μ from $(LH^{\beta,n}_{a,\mu,\alpha,m} ; T^{\beta,n}_{a,\mu,\alpha,m})$ into $(LH^{2\alpha,(2e)^{2\alpha}m^2}_{a,\mu,\alpha+\beta,e^{2\alpha}mn} ; T^{2\alpha,(2e)^{2\alpha}m^2}_{a,\mu,\alpha+\beta,e^{2\alpha}mn})$ is a continuous linear mapping.

PROOF : For the well defined character of the map lh_μ, we first prove that $\Phi \in LH^{2\alpha,(2e)^{2\alpha}m^2}_{a,\mu,\alpha+\beta,e^{2\alpha}mn}$ if $\varphi \in LH^{\beta,n}_{a,\mu,\alpha,m}$.

Starting from the equation (*) of Theorem 5.9 and using the fact that $\varphi \in LH^{\beta,n}_{a,\mu,\alpha,m}$ we have

$$(*)\ E \le M\,[\gamma^{\mu}_{a,v+k+2q,k,1}(\varphi) + \gamma^{\mu}_{a,v+k+2q+2+k,1}(\varphi)\,]$$

$$\le M\,[\{ C_{\delta\eta}\,(m+\delta)^{v+k+2q}(v+k+2q)^{(v+k+2q)\alpha}\,.$$

$$(n+\eta)^k\, k^{k\beta}\} + \{C_{\delta\eta}(m+\delta)^{v+k+2q+2}\,.$$

$$(v+k+2q+2)^{(v+k+2q+2)\alpha}\,(n+\eta)^k\, k^{k\beta}\}]$$

where M is some constant. Let us estimate

$$C_{\delta\eta}\,(m+\delta)^{v+k+2q}(v+k+2q)^{(v+k+2q)\alpha}\,(n+\eta)^k\, k^{k\beta}$$

$$= C_{\delta\eta}(n+\eta)^k\, k^{k\beta}(m+\delta)^k\,(m+\delta)^{2q}(m+\delta)^{v+2}.$$

$$(v+k)^{(v+k)\alpha}\left\{1+\frac{2q}{v+k}\right\}^{(v+k)\alpha}(v+k+2q)^{2q\alpha}$$

$$\leq C_{\delta\eta}(m+\delta)^k\,(n+\eta)^k\,(m+\delta)^{2q}\,k^{k\beta}\,(m+\delta)^{v+2}.$$

$$(v+k)^{v\alpha}(v+k)^{k\alpha}\,e^{2q\alpha}\,(2q)^{2q\alpha}\,e^{(v+k)\alpha}$$

$$\leq C_1\cdot C_{\delta\eta}\{mn+m\eta+n\delta+\delta\eta\}^k\,\{m^2+\delta_1\}^q\,k^{k(\alpha+\beta)}$$

$$e^{k\alpha}\,e^{v\alpha}\,e^{(v+k)\alpha}\,e^{2q\alpha}\,(2q)^{2q\alpha}$$

$$\leq C_2\cdot C_{\delta\eta}(e^{2\alpha}mn+\delta_2)^k\,\{m^2(2e)^{2\alpha}+\delta_3\}^q\,k^{k(\alpha+\beta)}\cdot q^{q(2\alpha)}.$$

Similarly, we have

$$C_{\delta\eta}(m+\delta)^{v+k+2q+2}(v+k+2q+2)^{(v+k+2q+2)\alpha}(n+\eta)^k\,k^{k\beta}$$

$$\leq C_3C_{\delta\eta}(e^{2\alpha}mn+\delta_2)^k\,\{m^2(2e)^{2\alpha}+\delta_3\}^q\,k^{k(\alpha+\beta)}\cdot q^{q(2\alpha)}$$

where C_1, C_2, C_3 are constants. Hence

$$E \leq C'_{\delta\eta}\,(e^{2\alpha}mn+\delta_2)^k\,\{m^2(2e)^{2\alpha}+\delta_3\}^q\,k^{k(\alpha+\beta)}\,q^{q(2\alpha)}$$

consequently, $\Phi \in LH^{2\alpha,\,(2e)^{2\alpha}m^2}_{a,\,\mu,\,(\alpha+\beta),\,e^{2\alpha}\cdot mn}$

Also, linearity of the map is straight forward and continuity follows from (*). #

6. SPACES OF LAPLACE-HANKEL TRANSFORMS:

This section is devoted to the study of several $\widetilde{LH}$-spaces which are vector spaces containing the Laplace-Hankel transforms

of members of LH-spaces studied in Chapter 3. The study we take up hereafter is similar in nature to that of Chapter 3. To be precise, let us recall Definitions 5.1 and 5.3 and introduce.

6.1 <u>THE SPACE</u> $\widetilde{LH}_{a,\mu,\alpha}$: is defined as

$$\widetilde{LH}_{a,\mu,\alpha} = \{ \Phi : \Phi = 1h_\mu \varphi, \varphi \in LH_{a,\mu,\alpha} \}.$$

The locally convex topology, denoted by $\widetilde{T}_{a,\mu,\alpha}$ on $\widetilde{LH}_{a,\mu,\alpha}$, is generated by the family of seminorms $\{ \lambda_{a,k,q,1} \}_{k,q,1=o}^{\infty}$ where

$$\lambda^{\mu}_{a,k,q,1} \Phi = \gamma^{\mu}_{a,k,q,1} \varphi, \varphi \in LH_{a,\mu,\alpha}.$$

6.2 <u>THE SPACE</u> $\tilde{\check{LH}}_{a,\mu,\alpha}$: It is given by

$$\tilde{\check{LH}}_{a,\mu,\alpha} = \{ \Phi : \Phi = 1h_\mu^- \varphi, \varphi \in \check{LH}_{a,\mu,\alpha} \}.$$

This space is equipped with the natural Hausdorff locally convex topology $\tilde{\check{T}}_{a,\mu,\alpha}$, generated by the family of seminorms $\{ \sigma^{\mu}_{a,k,q,1} \}_{k,q,1=o}^{\infty}$, where

$$\sigma^{\mu}_{a,k,q,1} \Phi = \rho^{\mu}_{a,k,q,1}(\varphi), \varphi \in \check{LH}_{a,\mu,\alpha}$$

In this case too, we have results analogous to Proposition 3.5.3 and 3.5.4.

PROPOSITION 6.3: For real number a, b with $a < b$, $\tilde{LH}_{b,\mu,\alpha} \subset \tilde{LH}_{a,\mu,\alpha}$ and $\tilde{T}_{a,\mu,\alpha} \mid \tilde{LH}_{b,\mu,\alpha} \subset \tilde{T}_{b,\mu,\alpha}$.

DEFINITION 6.4 : Let $\{a_n\}$ be an strictly decreasing sequence converging to r, $-\infty \leq r < \infty$. Set

$$\tilde{LH}(r,\mu,\alpha) = \bigcup_{n=1}^{\infty} \tilde{LH}_{a_n,\mu,\alpha}$$

and

$$\tilde{\check{LH}}(r,\mu,\alpha) = \bigcup_{n=1}^{\infty} \tilde{\check{LH}}_{a_n,\mu,\alpha}$$

where $\{\tilde{LH}_{a_n,\mu,\alpha}\}$ and $\{\tilde{\check{LH}}_{a_n,\mu,\alpha}\}$ are

increasing sequence of $\tilde{LH}$-spaces equipped with their usual topologies. Then the space $\tilde{LH}(r,\mu,\alpha)$ and $\tilde{\check{LH}}(r,\mu,\alpha)$ are inductive limits of these sequences when they are equipped with inductive limit topolgies defined with the help of injection maps from $\tilde{LH}_{a_n,\mu,\alpha} \to LH(r,\mu,\alpha)$ and $\tilde{\check{LH}}_{a_n,\mu,\alpha} \to \tilde{\check{LH}}(r,\mu,\alpha)$.

The spaces $\tilde{LH}_{a,\mu,\alpha}$ and $\tilde{\check{LH}}_{a,\mu,\alpha}$ are essentially the same in topological sense, for we have

THEOREM 6.5: Corresponding to φ in $LH_{a,\mu,\alpha}$ let $\Phi \in \tilde{LH}_{a,\mu,\alpha}$. Then $\check{\Phi} = lh^{-}_{\mu} \check{\varphi}$ belongs to $\check{LH}_{a,\mu,\alpha}$ and the mapping $S : (\tilde{LH}_{a,\mu,\alpha}; \tilde{T}_{a,\mu,\alpha}) \to (\tilde{\check{LH}}_{a,\mu,\alpha}, \tilde{\check{T}}_{a,\mu,\alpha})$ defined by $S(\Phi) = \check{\Phi}$ is a topological isomorphism for $Re(s) \in (-a,a)$.

PROOF : Since the existence of the integral

$$\int_0^{\infty} \int_0^{\infty} \exp(-st)\sqrt{xy}\, J_{\mu}(xy)\, \varphi(t,x)\, dxdt$$

is equivalent to the existence of

$$\int_{-\infty}^{0} \int_{0}^{\infty} \exp(-st) \sqrt{xy}\, J_{\mu}(xy)\, \varphi(-t,x)\,dx\,dt,$$

first part of the theorem follows. Hence the mapping S is well defined. Clearly, S is linear, one-one and onto. The continuity of S follows from Theorem 3.4.5 as the use of this result yields

$$\begin{aligned} \sigma^{\mu}_{a,k,q,l}(S\,\Phi) &= \rho^{\mu}_{a,k,q,l}(\check{\varphi}) \\ &= \gamma^{\mu}_{a,k,q,l}(\varphi) \\ &= \lambda^{\mu}_{a,k,q,l}(\Phi). \end{aligned}$$

Since the inverse mapping S^{-1} of S is given by $S^{-1}(\check{\Phi}) = \Phi$, the result is established.

CHAPTER - 5

IMPULSIVE DISTRIBUTIONS

CONTENTS

1. INTRODUCTION :

In this chapter, we confine our attention to δ-distributions, and their derivatives ; in otherwords, we concentrate on the study of impulsive distributions (cf. Definition 1.5.5). Indeed, we construct these distributions by extracting finite part from divergent integrals, use change of independent variable to define them and finally find Laplace-Hankel transforms of some of these distributions.

2. CONSTRUCTION :

As mentioned earlier, we show in this section that the finite part of a divergent integral obtained from the product of a regular and a delta distribution, defines an impulsive distribution. However, for such a construction the following lemma plays a key-role, for which we need recall notation δ_n, ρ and H_n from 1.5.3, 1.5.2 and 1.5.3.

LEMMA 2.1: For $t, x \in \mathbb{R}$ and $p, q = 0, 1, \ldots$ we have

$$\text{(i)} \quad \int_t^{1/n} x^p \, \delta_n^q(x)\,dx = \sum_{i=0}^{p} i\,!\,(-1)^{i+1}\, t^{p-i}\, \delta_n^{q-1-i}(t), \quad q > p$$

$$\text{(ii)} \quad \int_t^{1/n} x^p \, \delta_n^q(x)\,dx = \sum_{i=0}^{q} i\,!\,(-1)^{i+1}\, t^{q-i}\, \delta_n^{q-1-i}(t) + (-1)^q\, q\,!\,[\,1-H_n(t)\,], \quad q = p$$

PROOF : (i) Repeated use of integration by parts and the fact that supp $\delta_n \subset [-\frac{1}{n}, \frac{1}{n}]$ yield

$$\int_t^{1/n} x^p \delta_n^q(x)\,dx = \left[x^p \delta_n^{q-1}(x)\right]_t^{1/n} - p\int_t^{1/n} x^{p-1}\delta_n^{q-1}(x)\,dx$$

$$= -t^p \delta_n^{q-1}(t) - p\int_t^{1/n} x^{p-1}\delta_n^{q-1}(x)\,dx$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(*) \qquad = -t^p \delta_n^{q-1}(t) + \cdots + (-1)^p \int_t^{1/n} p!\,\delta_n^{q-p}(x)\,dx$$

$$= \sum_{i=0}^{p} i!\,(-1)^{i+1}\, t^{p-i}\,\delta_n^{q-1-i}(t)$$

which proves (i).

(ii) From (*), we have for $q = p$

$$\int_t^{1/n} x^p \delta_n^q = -t^q \delta_n^{q-1}(t) + \cdots + (-1)^q \int_t^{1/n} q!\,\delta_n(x)\,dx$$

$$= (-t)^q \delta_n^{q-1}(t) + \cdots + (-1)^q q!\left[H_n(x)\right]_t^{1/n}$$

(cf. Theorem 1.5.3)

$$= \sum_{i=0}^{q} i!\,(-1)^{i+1}\, t^{q-i}\,\delta_n^{q-1-i}(t) + (-1)^q q!\left[1 - H_n(t)\right]$$

This completes the proof. #

We are now set to prove the main result of this section, where in contrast to Definition 1.5.6, we give meaning to the product $(T_f \cdot \delta)$ when the sequence $\{f_n \cdot \delta_n\}$ diverges. However, in its statement and proof, we need two symbols F_p and Pf well known in the literature (cf. [98] p.41) with the names finite part of ...... and Pseudofunction.

THEOREM 2.2 : For given r, $0 \leq r < \infty$ let $T_{(x^r_+)}$ be the distribution defined by the function

$$x^r_+ = \begin{cases} x^r , & x > 0 \\ 0 , & x \leq 0 \end{cases}$$

Then for $m \geq 1$, the finite part of the product $\{T_{(x^r_+)} \cdot \delta^{r+2m-1}\}$ gives rise to an impulsive distribution equal to $\frac{(-1)^r (r+2m-1)!}{2(2m-1)!} \delta^{2m-1}$, denoted by $Pf\{T_{(x^r_+)} \cdot \delta^{r+2m-1}\}$.

PROOF : Recalling Definition 1.5.6, we have for x^r_+ ,

$$\begin{aligned} (x^r_+)_n &= \int_{-1/n}^{1/n} (x-t)^r \, \delta_n(t) \, dt \\ &= \int_{-1/n}^{x} (x-t)^r \, \delta_n(t) \, dt \end{aligned}$$

since $(x-t)^r = 0$ for $t \geq x$; and also

$$\begin{aligned} \langle T_{(x^r_+)} \cdot \delta^{r+2m-1} , \varphi \rangle &= \lim_{n \to \infty} \int_{-1/n}^{1/n} (x^r_+)_n \ \delta_n^{r+2m-1}(x) \, \varphi(x) \, dx \\ &= \lim_{n \to \infty} \int_{-1/n}^{1/n} \Big[\int_{-1/n}^{x} (x-t)^r \, \delta_n(t) \, dt \Big] \cdot \\ &\quad \cdot \delta_n^{r+2m-1}(x) \, \varphi(x) \, dx \\ &= \lim_{n \to \infty} \int_{-1/n}^{1/n} \delta_n(t) \, \Big[\int_{t}^{1/n} (x-t)^r \cdot \\ &\quad \cdot \delta_n^{r+2m-1}(x) \, \varphi(x) \, dx \Big] \, dt \end{aligned}$$

by changing the order of integration. Using Taylor's Theorem for expanding $\varphi(x)$, and collecting even and odd powers of x except the last two terms, we obtain four integrals :

$$I_1 = \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} (x-t)^r \delta_n^{r+2m-1}(x)$$

$$\cdot\{ \varphi(o) + \frac{x^2}{2!} \varphi''(o) + \cdots + \frac{x^{2m-2}}{(2m-2)!} \varphi^{2m-2}(o)\} dxdt$$

$$I_2 = \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} (x-t)^r \delta_n^{r+2m-1}(x) \cdot$$

$$\cdot\{ x\varphi'(o) + \cdots + \frac{x^{2m-3} \varphi^{2m-3}(o)}{(2m-3)!} \} dxdt$$

$$I_3 = \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} (x-t)^r \delta_n^{r+2m-1}(x) \quad \frac{x^{2m-1}}{(2m-1)!} \cdot \varphi^{2m-1}(o) dxdt$$

$$I_4 = \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} (x-t)^r \delta_n^{r+2m-1}(x) \cdot \frac{x^{2m}}{(2m)!} \varphi^{2m}(hx) dxdt$$

such that

$$(*) \quad \langle T_{(x_+^r)} \cdot \delta^{r+2m-1}, \varphi \rangle = \lim_{n\to\infty} [I_1 + I_2 + I_3 + I_4].$$

We now evaluate these integrals one by one.

Evaluation of I_1 : Observe that

$$I_1 = \sum_{j=0}^{r} \sum_{k=0}^{m-1} a_{jk} \int_{-1/n}^{1/n} \delta_n(t) \cdot t^j \int_t^{1/n} x^{r+2k-j} \cdot$$

$$\delta_n^{r+2m-1}(x) \; dxdt$$

where $a_{jk} = \binom{r}{j} \dfrac{(-1)^j \varphi^{2k}(o)}{(2k)!}$, $j = 0,1,..,r$, $k = 0,1,..,m-$ $m-1$. Using Lemma 2.1(1) for the integral $\int_t^{1/n} x^{r+2k-j} \cdot \delta_n^{r+2m-1}(x)dx$, we get

$$I_1 = \sum_{j=0}^{r} \sum_{k=0}^{m-1} \sum_{i=0}^{r+2k-j} b_{ijk} \int_{-1/n}^{1/n} t^{r+2k-i} \delta_n(t) \delta_n^{r+2m-2-i}(t)dt$$

where $b_{ijk} = (-1)^{i+1}(i)!\, a_{jk}$, $i = 0,1,..,r+2k-j$, $j = 0,1,..,r$, $k = 0,1,..,m-1$.

Substituting $x = nt$, $\delta_n(t) = n\,\rho(nt)$, we obtain

$$I_1 = \sum_{j=0}^{r} \sum_{k=0}^{m-1} \sum_{i=0}^{r+2k-j} b_{ijk} \int_{-1}^{1} \frac{x^{r+2k-i}}{n^{r+2k-i}} \cdot n\,\rho(x) \cdot$$

$$\cdot n^{r+2m-2-i+1} \rho^{r+2m-2-i}(x) \frac{dx}{n}$$

$$= \sum_{j=0}^{r} \sum_{k=0}^{m-1} \sum_{i=0}^{r+2k-j} b_{ijk}\, n^{2m-2k-1} \int_{-1}^{1} x^{r+2k-i} \rho(x) \cdot \rho^{r+2m-2-i}(x)dx$$

$\to \infty$ as $n \to \infty$.

Hence I_1 is a divergent integral for $n \to \infty$.

Evaluation of I_2 : For $a_{jk} = \binom{r}{j} \dfrac{(-1)^j \varphi^{2k-1}(o)}{(2k-1)!}$, $j = 0,..,r$, $k = 0,1,...,m-1$, we have

$$I_2 = \sum_{j=0}^{r} \sum_{k=1}^{m-1} a_{jk} \int_{-1/n}^{1/n} \delta_n(t)t^j \cdot \int_t^{1/n} x^{r+2k-j-1} \cdot \delta_n^{r+2m-1}(x)\,dxdt$$

Again applying Lemma 2.1(i),

$$I_2 = \sum_{j=0}^{r} \sum_{k=1}^{m-1} a_{jk} \int_{-1/n}^{1/n} \delta_n(t)\ t^j . \sum_{i=0}^{r+2k-j-1} (i)!\ (-1)^{i+1} .$$

$$.t^{r+2k-j-1-i}\ \delta_n^{r+2m-1-1-i}(t)\,dt$$

$$= \sum_{j=0}^{r} \sum_{k=1}^{m-1} \sum_{i=0}^{r+2k-j-1} b_{ijk} \int_{-1/n}^{1/n} t^{r+2k-1-i}\ \delta_n(t) .$$

$$.\ \delta_n^{r+2m-2-i}\ (t)\,dt$$

where $b_{ijk} = a_{jk}(i)!\ (-1)^{i+1}$, $i = 0,1,..,r+2k-j-1$, $j = 0,1,..,r$, $k = 1,2,..,m-1$. Since $\delta_n^p(t)$ is an even or odd function according as p is even or odd, the function $t^{r+2k-1-i}\ \delta_n^{r+2m-2-i}(t)$ is an odd function of t for any value of i,k and r. Hence

$$I_2 = 0$$

Evaluation of I_3 : Note that

$$I_3 = \sum_{j=0}^{r} \binom{r}{j} \frac{(-1)^j\ \varphi^{2m-1}(o)}{(2m-1)} \int_{-1/n}^{1/n} \delta_n(t)\ t^j .$$

$$.\int_t^{1/n} x^{r+2m-j-1}\ \delta_n^{r+2m-1}(x)\,dxdt$$

$$= \frac{\varphi^{2m-1}(o)}{(2m-1)!} \int_{-1/n}^{1/n} \delta_n(t) . \int_t^{1/n} x^{r+2m-1}\ \delta_n^{r+2m-1}\ dxdt$$

$$+ \sum_{j=1}^{r} \binom{r}{j} \frac{(-1)^j\ \varphi^{2m-1}(o)}{(2m-1)!} \int_{-1/n}^{1/n} \delta_n(t)\,t^j \int_t^{1/n} x^{r+2m-j-1} .$$

$$.\,\delta_n^{r+2m-1}(x)\,dxdt$$

Applying Lemma 2.1 (i) and (ii) for the second and the first integral respectively, we get

$$I_3 = \frac{\varphi^{2m-1}(o)}{(2m-1)!}\Big[\Big\{ \int_{-1/n}^{1/n} \delta_n(t) \Big(\sum_{i=0}^{r+2m-1} (i)!(-1)^{i+1}\, t^{r+2m-1-i}$$

$$\delta_n^{r+2m-2-i}(t) + (-1)^{r+2m-1}(r+2m-1)!\ \{1-H_n(t)\}\Big) dt\Big\}$$

$$+ \Big\{ \sum_{j=1}^{r} \binom{r}{j} (-1)^j \int_{-1/n}^{1/n} \delta_n(t)\ t^j \sum_{i=0}^{r+2m-j-1} (i)!$$

$$.(-1)^{i+1}\, t^{r+2m-j-1-i}\ \delta_n^{r+2m-2-i}(t)\,dt\Big\} \Big]$$

$$= (I_{31}+I_{32}+I_{33})$$

where

$$I_{31} = \frac{\varphi^{2m-1}(o)}{(2m-1)} \sum_{i=0}^{r+2m-1} (i)!\,(-1)^{i+1} \int_{-1/n}^{1/n} t^{r+2m-1-i}\ \delta_n(t)\,.$$

$$.\ \delta_n^{r+2m-2-i}(t)\,dt$$

$$I_{32} = \frac{\varphi^{2m-1}(O)}{(2m-1)!} (-1)^{r+2m-1}(r+2m-1)! \int_{-1/n}^{1/n} \delta_n(t)\{1-H_n(t)\}\,dt$$

$$I_{33} = \frac{\varphi^{2m-1}(o)}{(2m-1)!} \sum_{j=1}^{r} \binom{r}{j} (-1)^j \int_{-1/n}^{1/n} \delta_n(t)\ t^j \sum_{i=0}^{r+2m-j-1} (i)!$$

$$.(-1)^{i+1}\, t^{r+2m-j-1-i}\ \delta_n^{r+2m-2-i}(t)\,dt,$$

by following the same arguments as in the evaluation of I_2

for the function $t^{r+2m-1-i}$ $\delta_n^{r+2m-2-i}(t)$ and $t^{r+2m-1-i}$. $\delta_n^{r+2m-2-i}(t)$, we infer

$$I_{31} = 0, \ I_{33} = 0.$$

Let us therefore consider

$$I_{32} = \frac{(-1)^{r+2m-1}(r+2m-1)!}{(2m-1)!}\varphi^{2m-1}(o)\left[\int_{-1/n}^{1/n} \delta_n(t)\,dt - \int_{-1/n}^{1/n} H_n(t)\ \delta_n(t)\,dt\right].$$

Since $\int_{-1/n}^{1/n} \delta_n(t)\,dt = 1$

and

$$\int_{-1/n}^{1/n} H_n\ \delta_n(t)\,dt = \int_{-1/n}^{1/n} H_n \cdot H_n'(t)\,dt = \left\{\frac{(H_n(t))^2}{2}\right\}_{-1/n}^{1/n} = 0,$$

We obtain the value of I_{32} as

$$I_{32} = \frac{(-1)^{r+1}(r+2m-1)!}{2(2m-1)!}\ \varphi^{2m-1}(o).$$

Therefore

$$I_3 = I_{32} = \frac{(-1)^{r+1}(r+2m-1)!}{2(2m-1)!}\ \varphi^{2m-1}(o).$$

Evaluation of I_4 : To obtain the estimate of this integral, consider

$$|I_4| = \left| \sum_{j=0}^{r} \binom{r}{j} \frac{(-1)^j}{(2m)!} \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} x^{r-j}\, t^j. \right.$$

$$\left. \delta_n^{r+2m-1}(x)\, x^{2m}\, \varphi^{2m}(hx)\, dx dt \right|$$

Writing $M_j = \binom{r}{j} \frac{(-1)^j}{(2m)!} \sup_{(-1/n,1/n)} |\varphi^{2m}(hx)|$ and using the fact that $|x| \leq 1/n$, we have

$$|I_4| \leq \sum_{j=0}^{r} \frac{M_j}{n^{r+2m}} \left| \int_{-1/n}^{1/n} \delta_n(t) \left[\int_t^{1/n} \delta_n^{r+2m-1}(x)\, dx \right] dt \right|$$

Since $\delta_n(t) = n\,\rho(nt)$,

$$|I_4| \leq \sum_{j=0}^{r} \frac{M_j}{n^{r+2m}} \left| \int_{-1/n}^{1/n} n\,\rho(nt)\, n^{r+2m-1}\, \rho^{r+2m-2}(nt)\, dt \right|$$

$$= \sum_{j=0}^{r} \frac{M_j}{n^{r+2m}} \left| \int_{-1}^{1} n^{r+2m}\, \rho(x)\, \rho^{r+2m-2}(x)\, \frac{dx}{n} \right|$$

where $x = nt$. Observe that

$$\sum_{j=0}^{r} M_j \left| \int_{-1}^{1} \rho(x)\, \rho^{r+2m-2}(x)\, dx \right| \leq K,$$

for some finite positive constant K. Hence

$$|I_4| \leq \frac{K}{n} \to 0 \text{ as } n \to \infty .$$

Coming back to the equation (*), we have seen that its right hand side tends to ∞ because of the integral I_1, that is, I_1 contributes to the divergent part of

$< T_{x_+^r} \cdot \delta^{r+2m-1}, \varphi >$. Neglecting this divergent part and denoting the finite part of $<T_{x_+^r}, \delta^{r+2m-1}, \varphi>$ by $F_p \equiv F_p < T_{x_+^r} \cdot \delta^{r+2m-1}, \varphi >$, we get

$$F_p < T_{x_+^r} \cdot \delta^{r+2m-1}, \varphi > = \frac{(-1)^{r+1}(r+2m-1)!}{2(2m-1)!} \varphi^{2m-1}(o)$$

We now show that F_p defines a continuous linear functional on $D(\mathbb{R})$.

Clearly F_p maps $D(\mathbb{R})$ into $\mathbb{K}$ and is linear. For showing continuity of F_p, consider any balanced compact subset K of $\mathbb{R}$. Then

$$|F_p < T_{x_+^r} \cdot \delta^{r+2m-1}, \varphi >|$$

$$= \frac{(r+2m-1)!}{2(2m-1)!} |\varphi^{2m-1}(o)|$$

$$\leq \frac{(r+2m-1)!}{2(2m-1)!} \sup_{x \in K} |\varphi^{2m-1}(o)|$$

yields the desired continuity by Proposition 1.3.2.

Since

$$< \frac{(-1)^r(r+2m-1)!}{2(2m-1)!} \delta^{2m-1}, \varphi >$$

$$= \frac{(-1)^{r+1}(r+2m-1)!}{2(2m-1)!}\, \varphi^{2m-1}(o)$$

$$= F_p < \{T_{x_+^r} \cdot \delta^{r+2m-1}\}, \varphi >$$

$$= < P_f\{T_{x_+^r} \cdot \delta^{r+2m-1}\}, \varphi >$$

in view of the notation introduced in the beginning

We get

$$Pf\{T_{x_+^r} \cdot \delta^{r+2m-1}\} = \frac{(-1)^r(r+2m-1)!}{2(2m-1)!}\, \delta^{2m-1}$$

This completes the proof.#

A change in power of δ yields

THEOREM 2.3 : For the distribution $T_{(x_+^r)}$ as defined in preceding theorem and $m \geq 1$, the distribution $Pf\{T_{x_+^r} \cdot \delta^{r+2m}\}$, exists and is equal to impulsive distribution $\frac{(-1)^r(r+2m)!}{2(2m)!}\, \delta^{2m}$.

PROOF : It is analogous to that of Theorem 2.2 and so omitted #

3. CHANGE OF INDEPENDENT VARIABLES :

In this section we use the method of change of independent variable to define several impulsive distributions. In this direction let us recall the definition 1.5.7 and the function

g which we consider to be define from $\mathbb{R}$ to $\mathbb{R}$. Then corresponding to the delta distribution δ in $D^*(\mathbb{R})$ we get a distribution $g^S\delta$ in $D^*(\mathbb{R})$ defined by

$$\begin{aligned} < g^S\delta\,,\ \varphi> &= <\delta,\ |h'|\cdot\varphi\, oh > \\ &= |h'(o)|\cdot\varphi\, oh(o) \end{aligned}$$

where $h = g^{-1}$. This definition does not hold good in case $h'(0)$ is unbounded, that is $|h'(o)| = \infty$.

A modification of this definition so as to include the latter case is contained in

DEFINITION 3.1 : Let g be an infinitly differentiable, bijective function on $\mathbb{R}$, h the inverse of g such that $h'(o) = \infty$ and $r \geq 0$.

In case $\lim\limits_{x \to 0} [(|h'(x)|\{\varphi\, oh(x)\})^{(r)}]$ exists or can be separated into two sums corresponding to finite and infinite part, we get a unique distribution $g^S\delta^r$ in $D^*(\mathbb{R})$ for the delta distribution δ^r defined as follows

$$< g^S\delta^r, \varphi > = F_p[\lim_{x \to 0} \{(-1)^r|h'(x)|\{\varphi\, oh(x)\}\}^{(r)}]$$

To have a better understanding of Definition 3.1, we illustrate the same in the form of

THEOREM 3.2 : For given $m \geq 1$, let $x = g(t) = t^{2m+1}$ and $\delta \in D^*(\mathbb{R})$. Then the distribution $g^S\delta$ exists and is equal to $\dfrac{\delta^{2m}}{(2m+1)!}$.

PROOF : In this case, we have $t = h(x) = x^{1/2m+1}$ and $r = 0$ in Definition 3.1. Therefore consider

$$\left| \frac{d}{dx} x^{1/2m+1} \right| \{ \varphi o h(x) \}$$

$$= \frac{\varphi(t)}{(2m+1) x^{2m/2m+1}}$$

$$= \Big[\frac{1}{(2m+1) x^{2m/2m+1}} \{ \varphi(o) + t \ \varphi'(0) + \cdots$$

$$+ \frac{t^{2m-1} \ \varphi^{2m-1}(o)}{(2m-1)!} + \frac{t^{2m} \ \varphi^{2m}(0)}{(2m)!}$$

$$+ \frac{t^{2m+1} \ \varphi^{2m+1}(\theta t)}{(2m+1)!} \}$$

by Taylors expansion for φ. Substituting for $t = x^{1/2m+1}$, we get

$$= [S_1 + S_2 + S_3]$$

where

$$S_1 = \frac{1}{(2m+1) x^{2m/2m+1}} \{ \varphi(0) + x^{1/2m+1} \varphi'(0) + \cdots$$

$$\cdots + \frac{x^{2m-1/2m+1} \ \varphi^{2m-1}(0)}{(2m-1)!} \}$$

$$S_2 = \frac{\varphi^{2m}(0)}{(2m+1) \ (2m)!}$$

$$S_3 = \frac{x^{1/2m+1}\ \varphi^{2m+1}\ (\theta x^{1/2m+1})}{(2m+1)(2m+1)!}$$

Observe that

$$\lim_{x \to 0} S_1 = \infty$$

$$\lim_{x \to 0} S_2 = \frac{\varphi^{2m}(0)}{(2m+1)!}$$

$$\lim_{x \to 0} S_3 = 0$$

Hence

$$F_p \left[\lim_{x \to 0} \left| \frac{d}{dx}\, x^{1/2m+1} \right| \{\varphi\, oh(x)\}\right] = \frac{\varphi^{2m}(0)}{(2m+1)!}$$

Therefore

$$< g^S\delta\ ,\ \varphi > = \frac{\varphi^{2m}(0)}{(2m+1)!}$$

$$= < \frac{1}{(2m+1)!}\ \delta^{2m}\ ,\ \varphi >$$

Consequently

$$g^S\delta = \frac{1}{(2m+1)!}\ \delta^{2m}\ .\ \#$$

THEOREM 3.3 : For g as in Theorem 3.2 and $\delta^r \in D^*(\mathbb{R})$ $(r \geq 1)$, $g^S\delta^r$ exist in $D^*(\mathbb{R})$ and is equal to $\frac{r!\,\delta^{2mr+2m+r}}{(2m+1)(2mr+r+2m)!}$.

PROOF : As in the proof of the preceding theorem, consider

$$(-1)^r \frac{d^r}{dx^r} \{ \frac{1}{(2m+1)\, x^{2m/2m+1}} \varphi\, (x^{1/2m+1}) \}$$

where $\frac{\varphi(t)}{t^{2m}} = \frac{1}{t^{2m}} \; [\varphi(0) + t\,\varphi'(0) + \ldots + t^{2mr+2m+r-1}$,

$$\frac{\varphi^{2mr+2m+r-1}(0)}{(2mr+2m+r-1)\,!} + \frac{t^{2mr+2m+r} \;\; \varphi^{2mr+2m+r}\;(0)}{(2mr+r+2m)\,!}$$

$$+ \frac{t^{2mr+2m+r+1} \;\; \varphi^{2mr+2m+r+1}\;(\theta t)}{(2mr+2m+r+1)\,!}$$

Hence converting t into x, we get

$$(-1)^r \frac{d^r}{dx^r} [\frac{1}{(2m+1)\, x^{2m/2m+1}} \;\; \varphi(x^{1/2m+1})] \;\; = S_1 + S_2 + S_3$$

where

$$S_1 = (-1)^r \frac{d^r}{dx^r} [\frac{1}{(2m+1)\, x^{2m/2m+1}} \{ \varphi\,(0) + \;\; x^{1/2m+1} \;\varphi'(0) + \ldots$$

$$+ \;\; \frac{x^{\frac{2mr+2m+r-1}{2m+1}} \;\; \varphi^{2mr+2m+r-1}\;(0)}{(2mr+2m+r-1)\,!} \}]$$

$$S_2 = (-1)^r \frac{d^r}{dx^r} [\frac{1}{(2m+1)\, x^{2m/2m+1}} \;,\; \frac{x^{\frac{2mr+2m+r}{2m+1}} \;\; \varphi^{2mr+2m+r}\;(0)}{(2mr+2m+r)!}]$$

$$S_3 = (-1)^r \;\; \frac{d^r}{dx^r} [\frac{1}{(2m+1)\, x^{2m/2m+1}} \;\; \frac{x^{\frac{2mr+2m+r+1}{2m+1}} \;\; \varphi^{2mr+2m+r+1}(0)}{(2mr+2m+r+1)!}]$$

Here

$$\lim_{x \to 0} S_1 = \infty$$

$$\lim_{x \to 0} S_2 = \frac{(-1)^r \, r! \, \varphi^{2mr+2m+r}(0)}{(2m+1)(2mr+2m+r)!}$$

$$\lim_{x \to 0} S_3 = 0$$

consequently

$$g^S \delta^r = \frac{r! \, \varphi^{2mr+2m+r}}{(2m+1)(2mr+2m+r)!} \ .\#$$

Analogously we can prove

THEOREM 3.4 : For $a > 0$, $g_1(t) = t^3 + a$ and $g_2(t) = t^{2m+1} + a$, following holds

(i) $\quad g_1^S \delta_a = \frac{1}{3!} \delta_a^2$

(ii) $\quad g_2^S \delta_a = \frac{1}{(2m+1)!} \delta_a^{2m}$

PROOF : Omitted.

4. LAPLACE-HANKEL TRANSFORMS :

For the work of this section which includes the study of Laplace-Hankel transforms of impulsive distributions we have been motivated by Definitions 1.6.1 and 1.6.2 in order to introduce the notion of Laplace-Hankel transforms for distributions. Accordingly, we have

DEFINITION 4.1 : A distributional Laplace-Hankel transform, denoted by lh_{μ}^{*} is a linear functional from $LH_{a,\mu,\alpha}^{*}$ to $\mathbb{K}$ defined by

$$(*) \qquad \langle lh_{\mu}^{*} f, \Phi \rangle = 2\pi i \langle f, \check{\varphi} \rangle ,$$

where $f \in LH_{a\mu,\alpha}^{*}$, $\check{\varphi} \in LH_{a,\mu,\alpha}$ and $\Phi = lh_{\mu}\, \varphi$.

NOTE : Observe that $(*)$ gives a well defined map lh_{μ}^{*} in view of the Theorems 3.4.5 and 4.6.5.

We now restrict our attention to finding lh_{μ}^{*} of the delta-distribution $\delta_{a,b}$, $a,b > 0$ and its derivatives where

$$\langle \delta_{a,b}, \varphi \rangle = \varphi(a,b).$$

We begin with

THEOREM 4.2 : Let $a,b > 0$ and $G(s,y) = \sqrt{by}\cdot J_{\mu}(by) \exp(-sa)$. Then

$$lh_{\mu}^{*}\{\delta_{a,b}\} = T_G .$$

PROOF : Substituting $f = \delta_{a,b}$ in Definition 4.1 $(*)$, we get

$$\begin{aligned} \langle lh_{\mu}^{*}\, \delta_{a,b}, \Phi \rangle &= 2\pi i \langle \delta_{a,b}, \check{\Phi} \rangle \\ &= 2\pi i\, \check{\varphi}(a,b) \\ &= 2\pi i\, \varphi(-a,b) \\ &= 2\pi i \cdot \frac{1}{2\pi i} \int_0^{\infty} \sqrt{by}\;\; J_{\mu}(by) \cdot \\ &\qquad \cdot \int_{\sigma - i\infty}^{\sigma + i\infty} e^{-as}\, \Phi(s,y)\, ds\, dy \end{aligned}$$

(cf. Definition 4.5.4). Hence

$$< 1h_{\mu}^{*}\ \delta_{a,b},\ \Phi > = < T_{\sqrt{by}\,\cdot\, J_{\mu}(by)\exp(-sa)},\ \Phi >$$

$$= < T_G\ ,\ \Phi >$$

Consequently

$$1h_{\mu}^{*}(\delta_{a,b}) = T_{\sqrt{by}\ J_{\mu}(by)\exp(-sa)}\ . \#$$

Recalling the notation of a derivative of distribution from Definition 1.5.4, we finally prove

<u>THEOREM</u> 4.3 : For the distribution $\delta_{a,b}$ with $a, b > 0$, and $F(s,y) = [\{\sqrt{by}\ J'_{\mu}(by) + \frac{\sqrt{y}}{2\sqrt{b}}\ J_{\mu}(by)\}\ e^{-as}]$

$$1h_{\mu}^{*}\{\partial_x\ \delta_{a,b}\} = -T_F$$

<u>PROOF</u> : From the definition 4.1, we have

$$< 1h_{\mu}^{*}(\partial_x\ \delta_{a,b}),\ \Phi > = 2\pi i < \partial_x\ \delta_{a,b}\ ,\check{\varphi} >$$

$$= -2\pi i < \delta_{a,b},\ \partial_x\check{\varphi} >$$

$$= -2\pi i\ [\frac{\partial}{\partial x}\ \check{\varphi}\ ]_{t=a,\, x=b}$$

$$= -2\pi i\ [\frac{\partial\varphi}{\partial x}\ ]_{t=-a,\, x=b}$$

$$= -2\pi i\ [\frac{\partial}{\partial x}\cdot\frac{1}{2\pi i}\int_0^{\infty}\sqrt{xy}\ J_{\mu}(xy)\,\cdot$$

$$\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}\ \Phi(s,y)\,ds\,dy]_{\substack{t=-a\\ x=b}}$$

Observe that

$$\frac{\partial}{\partial x}\left[\sqrt{xy}\, J_\mu(xy)\right] = \sqrt{xy}\, J'_\mu(xy) + \frac{\sqrt{y}}{2\sqrt{x}} J_\mu(xy). \quad \text{Hence}$$

$$\langle\, lh^*_\mu(\partial_x\, \delta_{a,b}), \Phi \,\rangle = -\langle\, T_{\{\sqrt{by}\, J'_\mu(by) + \frac{\sqrt{y}}{2\sqrt{b}} J_\mu(by)\}\{e^{-as}\}}, \Phi \,\rangle$$

$$= \langle\, T_F, \Phi \,\rangle .$$

Consequently

$$lh^*_\mu \partial_x(\delta_{a,b}) = T_F \;. \#$$

Analogously we get

$$lh^*_\mu(\partial_x^r\, \delta_{a,b}) = (-1)^r\, T_{\left\{\frac{\partial^r}{\partial x^r}(\sqrt{xy}\, J_\mu(xy)\exp(-st)\right\}_{\substack{x=b\\ t=-a}}}$$

and

$$lh^*_\mu(\partial_t^p\, \delta_{a,b}) = (-1)^p\, T_{(s)^p \sqrt{by}\; J_\mu(by)\exp(-as)} \,.$$

Since an impulsive distribution is a linear combination of a δ distribution and its derivatives, we can easily obtain the Laplace-Hankel transforms of impulsive distributions by using above results.

CHAPTER - 6

APPLICATIONS

CONTENTS

1. INTRODUCTION :

This chapter incorporates applications of our earlier study to finding the solution of a partial differential equation arising in propagation of heat, relative to two different sets of boundary conditions given in terms of generalized functions. For the sake of convenience, the equation is considered in cylindrical coordinates with second coordinate θ being kept constant. As we shall see that the application of Laplace-Hankel transform reduces the given partial differential equation into an ordinary one with constant coefficients. However, the study requires some preliminary results on differential operators N_μ and M_μ introduced in Section 4 of Chapter 4.

2. RESULTS ON M_μ AND N_μ :

Let us recall the operators N_μ and M_μ from Definitions 4.4.1 and 4.4.3 and Propositions 4.4.9 and 4.4.5. Then we prove

PROPOSITION 2.1 : For $2\mu+1 \geq 0$ and $\varphi \in LH_{a,\mu+1,m,\alpha}$

$$lh_\mu(M_\mu \varphi) = y\, lh_{\mu+1}(\varphi)$$

PROOF : Consider

$$lh_\mu \{M_\mu \varphi(t,x)\} = lh_\mu \{x^{-(\frac{2\mu+1}{2})} D_x (x^{\frac{2\mu+1}{2}} \varphi(t,x)\}$$

$$= \int_0^\infty \int_0^\infty e^{-st} \sqrt{xy}\, J_\mu(xy) \left[x^{-\left(\frac{2\mu+1}{2}\right)} \cdot D_x\{ x^{\frac{2\mu+1}{2}} \varphi(t,x) \} \right] dxdt$$

$$= \int_0^\infty \sqrt{y}\, e^{-st} \left[\int_0^\infty \{ x^{-\mu} J_\mu(xy) \} \cdot \{ D_x(x^{\frac{2\mu+1}{2}} \varphi(t,x) \} \right] dxdt$$

Now

$$\int_0^\infty x^{-\mu} J_\mu(xy) \cdot \{ D_x(x^{\frac{2\mu+1}{2}} \varphi(t,x)) \} dx$$

$$= \left[\{ x^{-\mu} J_\mu(xy)\ x^{\frac{2\mu+1}{2}} \varphi(t,x) \}_0^\infty \right.$$

$$\left. - \int_0^\infty D_x\{ x^{-\mu} J_\mu(xy) \} \cdot x^{\frac{2\mu+1}{2}} \varphi(t,x)\, dxdt \right]$$

$$= - \int_0^\infty D_x \{ x^{-\mu} J_\mu(xy) \} \cdot x^{\frac{2\mu+1}{2}} \varphi(t,x)\, dxdt$$

since $\varphi(t,x)$ is an exponentially decreasing function and $J_\mu(xy)\ \varphi(t,x)$ is bounded as $x \to 0$.

Therefore

$$lh_\mu\{ M_\mu \varphi(t,x) \} = - \int_0^\infty \sqrt{y}\, e^{-st} \left[\int_0^\infty -y\, x^{-\mu} J_{\mu+1}(xy) \cdot x^{\frac{2\mu+1}{2}} \varphi(t,x)\, dx \right] dt$$

by the result $D_x\{ x^{-\mu} J_\mu(xy) \} = -y\, x^{-\mu} J_{\mu+1}(xy)$ (cf. [51] , p.154).

Hence

$$lh_\mu\, (M_\mu \varphi) = \int_0^\infty y\, e^{-st} \left[\int_0^\infty \sqrt{xy}\ J_{\mu+1}(xy) \cdot \varphi(t,x)\, dx \right] dt$$

$$= y\ lh_{\mu+1}(\varphi).$$

PROPOSITION 2.2 : For $2\mu+1 \geq 0$ and $\varphi \in LH_{a,\mu,\alpha,m}$

$$lh_{\mu}(M_{\mu}N_{\mu}\,\varphi) = -y^2\ lh_{\mu}(\varphi)$$

PROOF : Let $\Psi(t,x) = N_{\mu}\varphi(t,x)$. Then

$$lh_{\mu}\{M_{\mu}N_{\mu}\varphi(t,x)\} = lh_{\mu}\{M_{\mu}\Psi(t,x)\}$$

$$= y\ lh_{\mu+1}\Psi(t,x)$$

by Proposition (2.1). Therefore

$$lh_{\mu}\{M_{\mu}N_{\mu}\varphi\} = y\ lh_{\mu+1}[N_{\mu}\varphi]$$

$$= y(-y\ lh_{\mu}\varphi)$$

by Lemma 4.4.5. Hence the result follows. #

3. HEAT PROPAGATION EQUATION:

As mentioned in the beginning of this chapter, we are now prepared to solve heat propagation equation :

$$(3.1)\quad \frac{\partial^2\omega}{\partial r^2} + \frac{1}{r}\frac{\partial\omega}{\partial r} + \frac{\partial^2\omega}{\partial z^2} = \frac{\partial\omega}{\partial t}$$

We first formulate the problems with boundary conditions and then seek their solutions. To begin with

PROBLEM 3.2 : To find a function $\omega \equiv \omega(t,r,z)$ on the domain $\{(t,r,z): 0 < t < \infty,\ 0 < r < \infty,\ 0 < z < 1\}$ which satisfies the equation (3.1) with the following boundary conditions.

(BC1) : T_v^z converges to T_f as $z \to 0+$ in the space

$(LH^{*}_{a,o,\alpha,m} ; \sigma(LH^{*}_{a,o,\alpha,m}, LH_{a,o,\alpha,m}))$ where $v = \sqrt{r}\,\omega$ and f is a Laplace-Hankel transformable function defined on $\{(t,r): 0 < t < \infty , 0 < r < \infty\}$.

(BC2) : T_v^z converges to T_g as $z \to 1$ in the space $(LH^{*}_{a,o,\alpha,m} ; \sigma(LH^{*}_{a,o,\alpha,m} , LH_{a,o,\alpha,m}))$ where $v = \sqrt{r}\,\omega$ and g is a Laplace-Hankel transformable function defined on $\{(t,r) : 0 < t < \infty , 0 < r < \infty\}$.

(BC3) : As $t \to 0+$, ω converges to zero uniformly on $0 < r < \infty , 0 < z < 1$.

SOLUTION : For solving the equation (3.1) , we make use of operators M_μ, N_μ and lh_μ, for $\mu = 0$.

Let us first note that

$$\begin{aligned} M_o N_o v &= M_o N_o (r^{1/2}\,\omega) \\ &= r^{-1/2} D_r\, r^{1/2} [r^{1/2} D_r\, r^{-1/2} (r^{1/2}\,\omega)] \\ &= r^{-1/2} D_r [r D_r \omega] \\ &= r^{-1/2} [r D_r^2 \omega + D_r \omega] \\ &= r^{1/2} \frac{\partial^2 \omega}{\partial r^2} + r^{-1/2} \frac{\partial \omega}{\partial r} \end{aligned}$$

Hence, a multiplication of the equation (3.1) by $r^{1/2}$ yields

$$M_o N_o v + r^{1/2} \frac{\partial^2 \omega}{\partial z^2} = r^{1/2} \frac{\partial \omega}{\partial t}$$

Further substituting $\omega = r^{-1/2} v$, we get

$$(3.3) \qquad M_o N_o v + \frac{\partial^2 v}{\partial z^2} = \frac{\partial v}{\partial t}$$

Operating $1h_o$ on both the sides of (3.3), we get

$$(3.4) \qquad 1h_o M_o N_o v + 1h_o \frac{\partial^2 v}{\partial z^2} = 1h_o \frac{\partial v}{\partial t}$$

But by Proposition (2.2) with $\mu = 0$, $1h_o M_o N_o v = -u^2 v$ where $1h_o\, v(t,r,z) = V(s,u,z)$; therefore (3.4) reduces to

$$-u^2 V + \frac{\partial^2 V}{\partial z^2} = 1h_o \left(\frac{\partial v}{\partial t}\right).$$

We now simplify the right hand side of the above equation. Consider

$$1h_o\left(\frac{\partial v}{\partial t}\right) = \int_0^\infty \int_0^\infty e^{-st}\sqrt{ru}\, J_o(ru)\, \frac{\partial v}{\partial t}\, dtdr\ ;$$

but $\int_0^\infty e^{-st} \frac{\partial v}{\partial t}\, dt = \{ e^{-st} v\}_0^\infty + s\int_0^\infty e^{-st}\, v\, dt.$

Since $e^{-st} v = 0$ at the lower limit by boundary condition (BC3) and it is zero at the upper limit because $e^{-st} \to 0$ as $t \to \infty$ for $Re(s) > 0$.

Hence

$$1h_o\left(\frac{\partial v}{\partial t}\right) = sV\ ,$$

so that we get from (3.4)

$$\frac{\partial^2 V}{\partial z^2} - u^2 V = sV$$

or $\quad [\, D^2-(u^2+s)\,]\, V = 0 \;,\; D \equiv \frac{\partial}{\partial z}$

As is known, the general solution of the above equation is given by

(3.5) $V(s,u,z) = A\, e^{\{-(\sqrt{u^2+s})z\}} + B\, e^{\{(\sqrt{u^2+s})z\}}$,

where A and B are constants relative to z ; that is, $A \equiv A(s,u)$, $B \equiv B(s,u)$. Let T_V^z, $T_{Ae^{\{-(\sqrt{(u^2+s)})z\}}}$ and $T_{Be^{\{(\sqrt{u^2+s})z\}}}$ denote the distributions generated by functions $V(s,u,z)$, $A(s,u)\, e^{\{-(\sqrt{u^2+s})z\}}$ and $B(s,u)\, e^{\{(\sqrt{u^2+s})z\}}$ respectively. Then from (3.5) we infer

(3.6) $\quad T_V^z = T_{A\, e^{\{-(\sqrt{u^2+s})z\}}} + T_{B\, e^{\{(\sqrt{u^2+s})z\}}}$

In order to find A and B we use (3.6) and the boundary conditions (BC1) and (BC2).

First of all observe that for given $z > 0$,

$$T_V^z = lh_o^* \, T_v^z$$

where lh_o^* is the Laplace-Hankel transform of the distribution T_v^z (cf. Definition 5.4.1). Now consider

$$\begin{aligned} \langle lh_o^*\, T_v^z - lh_o^* T_f, \Phi \rangle &= \langle\, lh_o^*\, [T_v^z - T_f\,], \quad \Phi \rangle \\ &= 2\pi i \,\langle\, T_v^z - T_f, \overset{\vee}{\Phi} \rangle \\ &\to 0 \text{ as } z \to 0+ \end{aligned}$$

by the given boundary condition (BC1). Hence

$$(3.7) \qquad lh_o^{*} T_V^z \to lh_o^{*} T_f \text{ as } z \to o+$$

Note that the distribution $lh_o^{*} T_f$ is the distribution generated by the function $F(s,u)$, where

$$F(s,u) = \int_o^\infty \int_o^\infty e^{-st} \sqrt{ru}\, J_o(ru)\, f(t,r)\, dtdr$$

Moreover T_V^z and T_F belong to $\overset{\sim}{\overset{V}{LH}}_{a,o,\alpha,m}$. Hence from (3.7)

$$T_V^z \to T_F \text{ as } z \to o+ \text{ in } \sigma(\overset{\sim}{\overset{V}{LH}}{}^{*}_{a,o,\alpha,m}\,;\ \overset{\sim}{\overset{V}{LH}}_{a,o,\alpha,m})$$

Consequently from (3.6) we get

$$(3.8) \qquad T_F = T_A + T_B \,.$$

If G is Laplace - Hankel transform of g, that is

$$G(s,u) = \int_o^\infty \int_o^\infty e^{-st} \sqrt{ru}\ \ J_o(ru)\ g(t,r)\ dtdr$$

then proceeding on similar lines and using the boundary condition (BC2), we get

$$(3.9) \qquad T_G = T_A\, e^{-(\sqrt{u^2+s})} + T_B\, e^{\sqrt{u^2+s}}$$

Here

$$T_A\, e^{(-\sqrt{u^2+s})} = e^{-(\sqrt{u^2+s})}\ T_A$$

and

$$T_B\, e^{\sqrt{u^2+s}} = e^{\sqrt{u^2+s}}\ T_B$$

Hence

$$(3.10) \qquad T_G = e^{-(\sqrt{u^2+s})} T_A + e^{(\sqrt{u^2+s})} T_B \,.$$

For determining constants A and B, we solve equation (3.8) and (3.10) and obtain

$$T_A = \frac{e^{\sqrt{s+u^2}}\, T_F - T_G}{e^{\sqrt{s+u^2}} - e^{-\sqrt{s+u^2}}} ;$$

$$T_B = \frac{T_G - e^{-(\sqrt{s+u^2})}\, T_F}{e^{\sqrt{s+u^2}} - e^{-\sqrt{s+u^2}}} .$$

These values of distributions T_A and T_B yield

$$A(s,u) = \frac{F(s,u) \cdot e^{\sqrt{s+u^2}} - G(s,u)}{e^{\sqrt{s+u^2}} - e^{-(\sqrt{s+u^2})}} ;$$

$$B(s,u) = \frac{G(s,u) - F(s,u) \cdot e^{-(\sqrt{s+u^2})}}{e^{\sqrt{s+u^2}} - e^{-(\sqrt{s+u^2})}}$$

Substitution of these constants A(s,u) and B(s,u) in (3.5) now completely determines V(s,u;z). In order to find $\omega(t,r,z)$, observe that $A\, e^{-(\sqrt{u^2+s})z}$ and $B\, e^{(\sqrt{u^2+s})z}$ are smooth functions of (s,u) for fixed $z > 0$ and hence we can apply the conventional inverse Laplace-Hankel transform to get

$$(3.11)\quad v(t,r,z) = \frac{1}{2\pi i}\int_0^\infty \sqrt{ur}\, J_0(ur) \int_{\sigma-\infty}^{\sigma+i\infty} e^{st}\, V(s,u,z)\, ds\, du$$

Consequently,

$$\omega(t,r,z) = r^{-1/2} v(t,r,z)$$

where $v(t,r,z)$ is given by (3.11). This completes the solution.

Next, enlarging the domain of z from $(o,1)$ to (o,∞) we consider

PROBLEM 3.12 : Find a function $\omega \equiv \omega(t,r,z)$ on the domain $\{(t,r,z) : 0 < t < \infty : 0 < r < \infty , 0 < z < \infty\}$, which satisfies the partial differential equation (3.1) with the following boundary conditions:

(BC1) T_v^z converges to T_f as $z \to 0+$ in the space $(LH^*_{a,o,\alpha,m} ; \sigma(LH^*_{a,o,\alpha,m} ; LH_{a,o,\alpha,m}))$ where $v = \sqrt{r}\,\omega$ and f is a Laplace - Hankel transformable function defined on $\{(t,r): 0 < t < \infty , 0 < r < \infty\}$.

(BC2) T_v^z converges to zero distribution as $z \to \infty$ in the space $(LH^*_{a,o,\alpha,m} ; \sigma(LH^*_{a,o,\alpha,m}, LH_{a,o,\alpha,m}))$ where $v = \sqrt{r}\,\omega$

(BC3) ω converges to zero uniformly on $0 < r < \infty$, $0 < z < \infty$ as $t \to 0+$.

SOLUTION : Proceeding as in the solution of Problem (3.2) we reduce the equation (3.1) into

$$\frac{\partial^2 V}{\partial z^2} - u^2 V = sV$$

and obtain

$$(3.13)\quad V(s,u,z) = A(s,u)\ e^{-(\sqrt{u^2+s})z} + B(s,u)\ e^{(\sqrt{u^2+s})z} .$$

Consequently

$$T_V^z = T_{A(s,u)}\ e^{-(\sqrt{u^2+s})z} + T_{B(s,u)}\, e^{(\sqrt{s+u^2})z}$$

$$= e^{\{-(\sqrt{u^2+s})z\}}\ T_{A(s,u)} + e^{(\sqrt{u^2+s})z}\ T_{B(s,u)}$$

where all the notation signify the same meaning as in Problem (3.2).

We get

$$T_{B(s,u)} \equiv 0.$$

As $Re(\sqrt{u^2+s}) > 0$, using boundary condition (BC2).

Hence

$$T_V^z = e^{-(\sqrt{u^2+s})z}\ T_{A(s,u)}$$

Now application of the boundary condition (BC1) gives

$$T_F = T_{A(s,u)}$$

where F is given by

$$F(s,u) = \int_0^\infty \int_0^\infty e^{-st} \sqrt{ru}\ J_o(ru)\ f(t,r)\ dtdr$$

Therefore, the values of $A(s,u)$ and $B(s,u)$ are given by

$$A(s,u) = F(s,u)$$

$$B(s,u) = 0.$$

Hence we have

$$V(s,u,z) = F(s,u)\ e^{\{-(\sqrt{u^2+s})z\}}.$$

Since $F(s,u)\ e^{-\{(\sqrt{u^2+s})z\}}$ is a smooth function of (s,u) for fixed $z > 0$, applying conventional inverse Laplace - Hankel transform to get

$$(3.14) \qquad v(t,r,z) = \frac{1}{2\pi i} \int_0^\infty \sqrt{ur}\ J_0(ur) \int_{\sigma-j\infty}^{\sigma+j\infty} e^{st} \cdot V(s,u,z)\ ds\ du$$

Consequently the solution ω of equation (3.1) is given by

$$(3.15) \qquad \omega(t,r,z) = r^{-1/2}\ v(t,r,z)$$

where $v(t,r,z)$ is given by (3.14).

Finally regarding the solution of Problem 3.12, we have

PROPOSITION 3.16 : The solution ω of (3.1) in Problem (3.12) satisfies

(i) $\omega(t,r,z)$ converges to zero pointwise in (t,z) as $r \to \infty$, $0 < t < \infty$, $0 < z < \infty$.

(ii) $\omega(t,r,z)$ remains finite at each point on $0 < t < \infty$, $0 < z < \infty$ as $r \to 0+$.

PROOF: (i) For its proof, let us recall the order property of $J_\mu(x)$ from [51, p.147], namely

$$(3.17) \qquad \sqrt{ur}\, J_\mu(ur) = O(1) \ , \ r \to \infty$$

(i) is now an easy consequence of (3.15) and the equation (3.17).

(ii) Here to prove the result we make use of the following order property of $J_\mu(x)$, namely

$$\sqrt{ur}\ \ J_\mu(ur) = O(r^{\mu+\frac{1}{2}}) \ , \ r \to 0+$$

(cf. [51], p.134) and equation (3.15).

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S.No.	Title	Journal
1.	The Delta functional and the change of variable.	Math.Student, 45(1), (1977), 55-59.
2.	The Laplace transform of generalized functions	Math.Student, 46(3), (1978), 208-217.
3.	On a general stieltjes transform of a class of generalized functions.	Journal of M.A.C.T., 2(1978), 57-64.
4.	Singular Products of distributions.	Math.Ann. 240(1979), 39-40.
5.	Some theorems on a distributional generalized Stieltjes transform.	Journal, Indian Math.Soc., 43 (1979), 249-259.
6.	Pseudo functions and singular products of distributions.	Indian J. pure appl. Math., 10(9), (1979), 1082-1091.
7.	On spaces of type $LH_{a,\mu}$ and their Laplace-Hankel transformation.	Indian J.pure app.Math., 10(12),(1979), 1532-1542.
8.	On a distributional generalized Stieltjes transform.	Read at 49th annual session of N.A.Sc. India, 1979.
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14.	On spaces of type $LH_{a\mu}$	Indian.J. pure app.Math.,15(1),(1984), 59-64.
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16.	The Delta distribution and the change of variable.	Math.Student (to appear).
17.	On a distributional generalized Laplace Transform.	Math.Student (to appear).
18.	Stieltjes transformable generalized functions.	Bulletin Mathematique (to appear).